Chapter 4
Equations of elasticity

Chapters 2 and 3 of these notes do not specifically concern with the elastic media, in fact they can be understood for a generic continuum and studied independently. In this section we shall combine the previous results in order to to investigate the response of elastic bodies under the action of forces.

A body is called elastic if it has the property of recovering its original shape when the forces which produce the deformations are removed. This property can be characterized mathematically by certain relationships connecting force and displacement, that are also called constitutive laws. In particular we will analyze the linear constitutive law as a generalization of the Hooke's law.

4.1 The material law

It was Robert Hooke[1] who in 1676 gave the first rough law of proportionality between forces and displacements for an elastic body. In order to understand the key features of elasticity, let us consider a thin rod with an initial cross section $\mathcal{A}_0$, which is subjected to a variable tensile force F. We suppose that the stress is distributed uniformly over the area $\mathcal{A}_0$ and the initial cross–sectional area stays constant. The stress is obtained by dividing the force at any stage by the area $\mathcal{A}_0$. So, $\sigma = F/\mathcal{A}_0$. The relationship between F and the axial strain ε is plotted in figure 4.1 on the next page.

Figure 4.1 shows that until the point P the relationship $\sigma - \varepsilon$

[1] Robert Hooke (July 18, 1635 Freshwater (Isle of Wight) - March 3, 1703 London) was an English scientist.

Source: http://turnbull.mcs.st-and.ac.uk/history/Biographies/Hooke.html.

Claudio Borri, Michele Betti, Enzo Marino (a cura di), *Lectures on Solid Mechanics*, ISBN 978-88-8453-854-3 (online), ISBN 978-88-8453-853-6 (print),

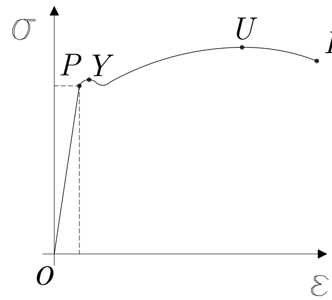

Figure 4.1: Hooke's law.

is nearly a straight line with the following equation

$$\sigma = E\varepsilon \tag{4.1}$$

where the constant of proportionality E is known as *modulus of elasticity* or *Young's modulus*.

The greatest stress that can be applied to the rod without producing a permanent deformation is called *elastic limit* of the material. When the force F is increased beyond this limit the material goes in the elastic-plastic field. Namely, firstly the material reaches the *yield–point* Y at which the rod suddenly stretches, then the material reaches the *ultimate stress* at U where it offers the maximum stress. If the elongation increases again both the cross sectional area $\mathcal{A}_0$ and the stress decrease until the rod breaks at B.

From now on we shall study only the *elastic range*.

4.1.1 Generalized Hooke's law

Here we want to extend the results of Hooke's law to a multidimensional state of stress and strain. So, in accordance with equation (4.1), let us write a linear relation

$$\sigma_{ij} = C_{ijhk}\varepsilon_{hk} \qquad i, j, h, k = 1, 2, 3 \tag{4.2}$$

The coefficients C_{ijhk} are independent from the position of the reference point in the continuous medium, in other words we require the homogeneity of the body, that means uniformity in structure and composition. It can also be shown that the elastic constants

C_{ijhk} are 81 components of a fourth order tensor which is termed *elasticity tensor*.

Since the stress tensor σ_{ij} is symmetric, an interchange of the first two indices in (4.2) does not alter its meaning. In addition to that, the symmetry of the strain tensor ensures also the symmetry of the last two indices, so that

$$C_{ijhk} = C_{jihk} \tag{4.3}$$

$$C_{ijhk} = C_{ijkh} \tag{4.4}$$

That means that the 3^4 components of C reduce to 36 independent constants. Let us show the expansion of a generic component of the stress tensor, that is

$$\begin{aligned} \sigma_{11} = & C_{1111}\varepsilon_{11} + C_{1112}\varepsilon_{12} + C_{1113}\varepsilon_{13} + \\ & C_{1121}\varepsilon_{21} + C_{1122}\varepsilon_{22} + C_{1123}\varepsilon_{23} + \\ & C_{1131}\varepsilon_{31} + C_{1132}\varepsilon_{32} + C_{1133}\varepsilon_{33} \end{aligned} \tag{4.5}$$

Equations (4.3) and (4.4) allow (4.5) to be rewritten as follows

$$\begin{aligned} \sigma_{11} = & C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + C_{1133}\varepsilon_{33} + \\ & 2C_{1112}\varepsilon_{12} + 2C_{1113}\varepsilon_{13} + 2C_{1123}\varepsilon_{23} \end{aligned}$$

Thus, the whole elastic matrix can be written as

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1112} & 2C_{1123} & 2C_{1131} \\ & C_{2222} & C_{2233} & 2C_{2212} & 2C_{2223} & 2C_{2231} \\ & & C_{3333} & 2C_{3312} & 2C_{3323} & 2C_{3331} \\ & & & 2C_{1212} & 2C_{1223} & 2C_{1231} \\ & \text{sym.} & & & 2C_{2323} & 2C_{2331} \\ & & & & & 2C_{3131} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{pmatrix}$$

which, making use of the symmetry relationships expressed in (4.3) and (4.4), simplifies as follows

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ & & c_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44} & c_{45} & c_{46} \\ & \text{sym.} & & & c_{55} & c_{56} \\ & & & & & c_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{pmatrix}$$

Later on, see equation (6.10), we will also introduce another symmetry condition that has been assumed in the above. Namely, the condition

$$C_{ijhk} = C_{hkij} \tag{4.6}$$

that further reduces the independent elastic constant from 36 to 21. So, the latter material equation represents the constitutive law for an *anisotropic* elastic material. However, most of the engineering materials have some symmetry properties which allow further reductions of the elastic constants.

The highest degree of symmetry leads to the so called *isotropic* material. We define an isotropic material an elastic continuum which has the same response in any direction, so that the elastic tensor is not influenced by any rotation of the references axes.

Let the elastic tensor be represented by C_{ijhk} with respect to the cartesian coordinate $\{x^i\}$ whose basis is $\mathcal{B} = \{\bar{e}_i\}$. With respect to a rotated system $\{x'^i\}$ with basis $\mathcal{B}' = \{\bar{e}'_i\}$ the elasticity tensor is C'_{ijhk}. By the definition of isotropic material, we expect that the elasticity tensor does not change. In order to show this, let us recall the transformation relations (1.36) on chapter 1. Here we are dealing with a Cartesian coordinate system, hence it does not matter if the indices are all subscripts. So, we have

$$\begin{aligned} C'_{ijhk} &= a'_{il}a'_{jm}C_{lmno}a_{oh}a_{nk} \\ &= a'_{il}a'_{jm}a'_{ho}a'_{kn}C_{lmno} \end{aligned} \tag{4.7}$$

but to ensure the immunity against the rotation of the reference system, we impose

$$C'_{ijhk} = C_{lmno} \tag{4.8}$$

that is only satisfied if the elasticity tensor assumes the following form

$$C_{lmno} = \lambda\delta_{lm}\delta_{no} + \mu\delta_{ln}\delta_{mo} + \kappa\delta_{lo}\delta_{mn} \tag{4.9}$$

where λ, μ, κ are elastic constants[2].

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[2]This can be proved by replacing equation (4.9) into (4.7), as follows

$$\begin{gathered} C'_{ijhk} = a'_{il}a'_{jm}a'_{ho}a'_{kn}\left(\lambda\delta_{lm}\delta_{no} + \mu\delta_{ln}\delta_{mo} + \kappa\delta_{lo}\delta_{mn}\right) = \\ = \lambda a'_{im}a'_{jm}a'_{ho}a'_{ko} + \mu a'_{in}a'_{jo}a'_{hn}a'_{ko} + \kappa a'_{io}a'_{jn}a'_{hn}a'_{ko} = \\ \lambda\delta_{ij}\delta_{hk} + \mu\delta_{ih}\delta_{jk} + \kappa\delta_{ik}\delta_{jh} \end{gathered}$$

that is exactly the expression (4.9). Note that we have used the identity $a'_{ps}a'_{qs} = \delta_{pq}$ provided by equations (1.21) and (1.24) on page 7.

In equations (4.3) and (4.4) we have already noticed the symmetry of C in relation to the two front and two back indices, let us show now that one more reduction is possible

$$C_{ijhk} = \lambda\delta_{ij}\delta_{hk} + \mu\delta_{ih}\delta_{jk} + \kappa\delta_{ik}\delta_{jh} \tag{4.10}$$
$$C_{ijkh} = \lambda\delta_{ij}\delta_{kh} + \mu\delta_{ik}\delta_{jh} + \kappa\delta_{ih}\delta_{jk} \tag{4.11}$$

where, subtracting term by term and considering the symmetry of the unit tensor δ_{ij}, equations (4.10) and (4.11) lead to the only possible condition

$$\begin{aligned}
\mu\left(\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}\right) + \kappa\left(\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk}\right) = 0 \Rightarrow \\
\mu\left(\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}\right) - \kappa\left(\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}\right) = 0 \Rightarrow \\
(\mu - \kappa)\left(\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}\right) = 0
\end{aligned} \tag{4.12}$$

which is only true if $(\mu - \kappa) = 0$. So, the relationship between κ and μ further reduces the number of elastic constants to 2. Namely, we have

$$C_{ijhk} = \lambda\delta_{ij}\delta_{hk} + \mu\left(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}\right) \tag{4.13}$$

The Hooke's law becomes

$$\begin{aligned}
\sigma_{ij} = C_{ijhk}\varepsilon_{hk} &= \lambda\delta_{ij}\delta_{hk}\varepsilon_{hk} + \mu\left(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}\right)\varepsilon_{hk} = \\
&= \cdots \\
&= \lambda\delta_{ij}\varepsilon_{hh} + 2\mu\varepsilon_{ij}
\end{aligned} \tag{4.14}$$

where we have used $\delta_{hk}\varepsilon_{hk} = \varepsilon_{hh} = \mathrm{tr}\varepsilon_{hk}$.

Equation (4.14) is the generalized form of Hooke's law, valid for homogeneous, isotropic, elastic bodies. λ and μ are called *Lamé constants*[3].

[3]Gabriel Lamé (July 22, 1795 Tours - May 1, 1870 Paris) was a French mathematician and engineer.

Source: http://turnbull.mcs.st-and.ac.uk/history/Mathematicians/Lame.html.

The trace of the stress tensor is readily computed by contracting the indices, so that

$$\sigma_{ii} = 3\lambda\varepsilon_{hh} + 2\mu\varepsilon_{ii} \Rightarrow \tag{4.15}$$

$$\sigma_{ii} = (2\mu + 3\lambda)\,\varepsilon_{hh} \Rightarrow \tag{4.16}$$

$$\varepsilon_{hh} = \frac{\sigma_{ii}}{(2\mu + 3\lambda)} \tag{4.17}$$

where we can put $\mathrm{tr}\sigma_{ij} = \sigma_{ii} = \Sigma$ and $\mathrm{tr}\varepsilon_{ij} = \varepsilon_{ii} = \Theta$.

The above expression (4.17) is useful if we solve (4.14) for ε_{ij}. In fact, we have

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu}\delta_{ij}\Theta \tag{4.18}$$

and in observance of (4.17) we obtain

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu\,(3\lambda + 2\mu)}\delta_{ij}\Sigma \tag{4.19}$$

Now, let us consider an axial state of stress. The stress tensor is

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form (4.19) we have

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{2\mu}\left(1 - \frac{\lambda}{(3\lambda + 2\mu)}\right)\sigma_{11} = \\ &= \cdots \\ &= \frac{\lambda - \mu}{\mu\,(3\lambda + 2\mu)}\sigma_{11} \qquad (4.20) \\ \varepsilon_{22} = \varepsilon_{33} &= -\frac{\lambda}{2\mu\,(3\lambda + 2\mu)}\sigma_{11} \qquad (4.21) \\ & \qquad (4.22) \end{aligned}$$

Let us define *Poisson's ratio* ν as follows

$$\nu = -\frac{\varepsilon_{11}}{\varepsilon_{22}} = -\frac{\varepsilon_{11}}{\varepsilon_{33}} = \frac{\lambda}{2\,(\mu + \lambda)} \tag{4.23}$$

	λ	$\mu \equiv G$	E	ν
λ, μ	-	-	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$
λ, ν	-	$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	-
μ, E	$\frac{\mu(E-2\mu)}{3\mu-E}$	-	-	$\frac{E-2\mu}{2\mu}$
μ, ν	$\frac{2\mu\nu}{1-2\nu}$	-	$2\mu\,(1+\nu)$	-
E, ν	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	-	-

Table 4.1: Relationships between the main elastic constants.

According to Hooke's law in the original form, see equation (4.1), we can see that

$$\frac{1}{E} = \frac{\lambda - \mu}{\mu\,(3\lambda + 2\mu)} \Rightarrow E = \frac{\mu\,(3\lambda + 2\mu)}{\lambda - \mu} \tag{4.24}$$

So, we have proved that Lamé constants can be replaced by E and ν which lead to writing the alternative expressions of the constitutive law

$$\varepsilon_{ij} = \frac{1}{E}\left((1+\nu)\,\sigma_{ij} - \nu\delta_{ij}\Sigma\right) \tag{4.25}$$

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\Theta\right) \tag{4.26}$$

Table 4.1 shows the relationships between elastic constants.

4.2 The linear elastic problem

In this section we are going to sum up equations and unknown quantities which define the classical linear elastic problem. Then we will estimate the distribution of stresses and strain as well as displacements at all points of the body when certain boundary conditions are given. Let us balance the unknowns and the equations, we have fifteen unknowns (6 stress components + 6 strain components + 3 displacement components) for all points in the continuous and just fifteen equations (6 equilibrium + 6 compatibility + 3 boundary conditions). So, for a given linear elastic body $\mathcal{V}$ we have

$$C = const. \tag{4.27}$$

$$\bar{b} = \bar{b}\,(p) \quad \forall p \in \mathcal{V} \tag{4.28}$$

$$\bar{f} = \hat{\bar{f}}\,(p) \quad \forall p \in \mathcal{S}_\sigma \tag{4.29}$$

$$\bar{u} = \hat{\bar{u}}\,(p) \quad \forall p \in \mathcal{S}_u \tag{4.30}$$

In order to solve the linear elastic problem we start from the known quantities (4.27) to (4.30), and through the following available equations

- compatibility equations

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad \text{on } \mathcal{V} \tag{4.31}$$

- equilibrium equations

$$\sigma_{ij,j} + b_i = 0 \quad \text{on } \mathcal{V} \tag{4.32}$$

- constitutive laws

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{ij}\right) \quad \text{on } \mathcal{V} \tag{4.33}$$

- boundary conditions

$$\sigma_{ij} n_j = \hat{f}_i \quad \text{on } \mathcal{S}_\sigma \tag{4.34}$$

$$u_i = \hat{u}_i \quad \text{on } \mathcal{S}_u \tag{4.35}$$

we will formulate two boundary–value problems.

4.2.1 Boundary value problem in terms of stresses

This first boundary value problem can be stated as follows:

> *Determine the distribution of stresses and displacements in the interior of an elastic body in equilibrium when the body forces are prescribed and the distribution of the forces acting on the surface of the body is known*[4].

[4]Sokolnikoff [1].

Following the above formulation, the procedure for solving the problem would suggest writing the available equations entirely in terms of stress. To this aim let us start from equation (2.75)

$$\varepsilon_{ij,hk} + \varepsilon_{hk,ij} - \varepsilon_{ih,jk} - \varepsilon_{jk,ih} = 0 \tag{4.36}$$

and consider the constitutive law (4.25), so that

$$\begin{aligned} &\frac{1+\nu}{E}\left(\sigma_{ij,hk} + \sigma_{hk,ij} - \sigma_{ih,jk} - \sigma_{jk,ih}\right) = \\ &= \frac{\nu}{E}\left(\delta_{ij}\sigma_{nn,hk} + \delta_{hk}\sigma_{nn,ij} - \delta_{ih}\sigma_{nn,jk} - \delta_{jk}\sigma_{nn,ih}\right) \end{aligned} \tag{4.37}$$

Equation (4.37) represents a set $3^4 = 81$ equations since all the four indices i, j, h, k run from 1 to 3. Not all of these equations are independent, indeed the system (4.37) contains only 6 independent equations. A first reduction of equations is due to the contraction $h = k$ that yields

$$\begin{aligned} &\sigma_{ij,kk} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} = \\ &= \frac{\nu}{1+\nu}\left(\delta_{ij}\sigma_{nn,kk} + \delta_{kk}\sigma_{nn,ij} - \delta_{ik}\sigma_{nn,jk} - \delta_{jk}\sigma_{nn,ik}\right) \end{aligned} \tag{4.38}$$

that, by denoting $\Sigma = \mathrm{tr}\sigma_{ij} = \sigma_{ii}$ and $\sigma_{ij,kk} = \nabla^2\sigma_{ij}$, becomes

$$\nabla^2\sigma_{ij} + \Sigma_{,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} = \frac{\nu}{1+\nu}\left(\delta_{ij}\nabla^2\Sigma + \nabla^2\Sigma_{ij}\right) \tag{4.39}$$

By virtue of the equilibrium equations (4.32), the above expression can be rewritten as follows

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\Sigma_{,ij} = -\left(b_{i,j} + b_{j,j} - \frac{\nu}{1+\nu}\delta_{ij}\nabla^2\Sigma\right) \tag{4.40}$$

which is a set of 6 independent equations.

Next, in order to express $\nabla^2\Sigma$ as a function of the body force $\bar{b}$, we put $h = i$ and $k = j$ in equation (4.37), so that, after a bit of algebra, we have

$$\begin{aligned} \sigma_{ij,ij} &= \nabla^2\Sigma - 2\frac{\nu}{1+\nu}\nabla^2\Sigma \\ &= \dots \\ &= \frac{1-\nu}{1+\nu}\nabla^2\Sigma \end{aligned} \tag{4.41}$$

and finally, by invoking the derivative of the equilibrium equation that provides the relationships $b_{i,i} = \sigma_{ij,ij}$, we get

$$\nabla^2 \Sigma = -\frac{1+\nu}{1-\nu} b_{i,i} \tag{4.42}$$

Now, going back to equation (4.40) and making use of the latter result, it is not a difficult task to obtain the following expression

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu} \Sigma_{,ij} = -\left(b_{i,j} + b_{j,i} + \frac{\nu}{1-\nu} \delta_{ij} \mathrm{div}\, \bar{b} \right) \tag{4.43}$$

Equations (4.43) were derived by *Michell*[5] in 1900 and by *Beltrami*[6] in the 1892 for the special case when the body forces are absent. Nevertheless, it is common to refer to equation (4.43) as *Beltrami-Michell* equations.

In case of missing or constant volume forces equation (4.43) assumes the straightforward form

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu} \Sigma_{ij} = 0 \tag{4.44}$$

4.2.2 Boundary value problem in terms of displacements

The second boundary value problem can be stated as follows:

Determine the distribution of stresses and displacements in the interior of an elastic body in equilibrium

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[5] John Henry Michell (October 26, 1863 - February 3, 1940) was an Australian mathematician.

Source:http://en.wikipedia.org/wiki.

[6] Eugenio Beltrami (November 16, 1835 Cremona - February 18, 1900 Rome) was an Italian mathematician.

Source: http://www-groups.dcs.st-and.ac.uk/ history/Biographies/Beltrami.html.

when the body forces are prescribed and the displacements of the points on the surface are prescribed functions[7]*.*

By replacing the constitutive law in the form of (4.14) into equilibrium equation, we obtain

$$\left(\lambda\delta_{ij}\varepsilon_{kk}\right)_{,j} + 2\mu\varepsilon_{ij,j} + b_i = 0 \tag{4.45}$$

that is

$$\lambda\varepsilon_{kk,i} + 2\mu\varepsilon_{ij,j} + b_i = 0 \tag{4.46}$$

and in accordance with the compatibility equations we have

$$\lambda u_{k,ki} + \mu\left(u_{i,jj} + u_{j,ij}\right) + b_i = 0 \tag{4.47}$$

$$\lambda u_{k,ki} + \mu\nabla^2 u_i + \mu u_{k,ik} + b_i = 0 \tag{4.48}$$

$$\left(\lambda + \mu\right) u_{k,ki} + \mu\nabla^2 u_i + b_i = 0 \tag{4.49}$$

that in the vectorial form reads

$$\left(\lambda + \mu\right) \operatorname{grad}\operatorname{div}\bar{u} + \mu\nabla^2\bar{u} + \bar{b} = 0 \tag{4.50}$$

Equation (4.49) (or equivalently equation (4.50)) is called *Lamé-Navier* equation and together with the boundary conditions expressed by equation (4.35) define the boundary problem inn terms of displacements.

Once the first boundary value problem has been solved, i.e. when the displacements are known, the state of strain and hence the state of stress can be found though equations (4.31) and (4.33), respectively.

Further attention should be focused on the case when body forces do not occur or they are constant. First, consider the divergence of equation (4.49)

$$\left(\lambda + \mu\right) u_{k,kii} + \mu\nabla^2 u_{i,i} + b_{i,i} = 0 \tag{4.51}$$

that yields

$$\lambda\nabla^2 u_{k,k} + 2\mu\nabla^2 u_{k,k} + b_{i,i} = \left(\lambda + 2\mu\right)\nabla^2 u_{k,k} + b_{i,i} = 0 \tag{4.52}$$

[7] Sokolnikoff [1].

which, under the hypothesis of $b_i = \text{const.}$, so that $b_{i,i} = 0$, gives

$$\nabla^2 u_{k,k} = \nabla^2 \Theta = 0 \tag{4.53}$$

where we have set $\Theta = \text{tr}\varepsilon_{ij} = \varepsilon_{ii}$.

Moreover, recalling (4.17) it is also proved that

$$\nabla^2 \sigma_{kk} = 0 \tag{4.54}$$

We can finally say that if the volume forces are constant, the boundary linear elastic problem in terms of displacements turns into a general boundary values problem of a biharmonic differential equation.

4.3 Constitutive equation for shell continuums

The Kirchhoff–Love hypothesis and the inextensibility of material fibers along $\bar{n}$ allows one to consider the shear stress components $N^{\xi\alpha}$ unrelated to strains, so that the constitutive problem can be solved through the plane stress model. Thus, components $N^{\xi\alpha}$ are found only by means of the equilibrium equations. The analytical derivation of the constitutive equations is beyond the scope of this book, so we will just present the final equations that will be used in the appendix A in order to solve some case studies. However, readers can find thorough discussions in [3] and [16].

Suppose a membrane state of stress, the constitutive equations are the following

$$\tilde{N}^{\alpha\beta} = D H^{\alpha\beta\lambda\mu} \alpha_{\lambda\mu} \tag{4.55}$$

$$M^{\alpha\beta} = B H^{\alpha\beta\lambda\mu} \omega_{\lambda\mu} \tag{4.56}$$

where

$$H^{\alpha\beta\lambda\mu} = \frac{1-\nu}{2}\left(g^{\alpha\lambda}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\lambda} + \frac{2\nu}{1-\nu}g^{\alpha\beta}g^{\lambda\mu}\right) \tag{4.57}$$

The fourth–order tensor $H^{\alpha\beta\lambda\mu}$ has the following symmetries

$$H^{\alpha\beta\lambda\mu} = H^{\beta\alpha\lambda\mu} = H^{\alpha\beta\mu\lambda} = H^{\lambda\mu\alpha\beta} \tag{4.58}$$

Finally, coefficients D and B are the in–plane and the bending stiffness, respectively, defined as

$$D = \frac{E(2\varepsilon)}{1-\nu^2} \tag{4.59}$$

$$B = \frac{E(2\varepsilon)^3}{12(1-\nu^2)} \tag{4.60}$$