4. First and second Gödel's theorems and related results

4.1. Definability and representability

We discussed the definability of the sets, so let's begin this section by talking in particular about the definability *of functions*, to come next to their *representability*, as in Gödel's original approach, when he proved that every computable function and computable set is representable in Robinson's Arithmetic Q. Roughly speaking, the notion of definability is semantic, given in terms of the notion of *truth*, while representability is a corresponding tighter syntactic notion. We will later see the close relationship between these two concepts. To our pourpose, it will be necessary to investigate the Σ_1 definability in the standard model. Saying that $\phi(x_0, ..., x_k)$ is Σ_1-definable e.g. in the standard model we mean therefore to say that *its graph*:

$$G_\phi = \{\langle x_0, ..., x_k, y\rangle | \phi(x_0, ..., x_k) \simeq y\}$$

is Σ_1-definable in it. We will see later the relationship with the notion of representability.

Theorem 67. *Consider the language of* PA*. The following holds:*

1. *If a set X is Δ_0-definable in the standard model, then X is primitive recursive.*
2. *X is Σ_1-definable in the standard model, iff X is computably enumerable*
3. *ϕ is Σ_1-definable in the standard model, iff ϕ is partial recursive.*

Proof. The proof is articulated in several passeges. As for 1., note that all terms of the language based on $+, \times, S, 0, \leq$ denote primitive recursive functions. In addition the bounded quantifiers do not get out of this level. Regarding point 2., we have already seen at p. 43. Finally 3. $\Rightarrow$, if $G_\phi = \{\langle x, z\rangle | \phi(x) \simeq z\}$ is Σ_1, then is r.e and therefore is the domain of a partial recursive function $\phi_e(x, z)$. Hence $\langle x, z\rangle \in G_\phi$ iff $\exists y T(e, x, z, y)$. Let therefore:

$$\phi(x) \simeq (\mu u.u = \langle z, y\rangle \wedge T(e, x, (u)_0, (u)_1))_1$$

The crucial direction is $\Leftarrow$: all partial recursive functions are Σ_1-definable in the language of PA. We associate to any function $\phi(x)$, a Σ_1-formula $\psi_\phi(x, y)$ defining its graph G_ϕ, that is to say:

$$\psi_\phi(x, y) \text{ is true iff } \langle x, y\rangle \in G_\phi \text{ iff } \phi(x) \simeq y$$

1. *Initial functions*:

Function	*Definition*
$Z(x) \simeq 0$	$(x = x \wedge y = 0)$
$S(x) \simeq x + 1$	$(y = S(x))$
$U^m_i(x_0, ..., x_n) = x_i$	$\bigwedge_{0 \leq j \leq n} x_j = x_j \wedge x_i = y$

(1)

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2. *Composition*: Suppose ψ_g, $\psi_{h_0}, ..., \psi_{h_k}$ let Σ_1-definitions of the functions $g, h_0, ..., h_k$ and let $\phi(x) \simeq g(h_0(x), ..., h_k(x))$.
   Let therefore $\psi_\phi(x, z)$ the Σ_1-formula:

$$\exists y_0 ... \exists y_k(\psi_{h_0}(x, y_0) \wedge ... \wedge \psi_{h_k}(x, y_k) \wedge \psi_g(y_0, ..., y_k, z))$$

3. *Minimization*: let $\phi(x) \simeq \mu z.g(x, z) \simeq 0$ where $\forall v < z(g(x, v) \downarrow \wedge g(x, v) \neq 0)$ and let ψ_g a Σ_1-definition of g; let then $\psi_\phi(x, z)$ the following Σ_1-formula:

$$\psi_g(x, z, \overline{0}) \wedge \forall v < z \exists w(\psi_g(x, v, w) \wedge \neg(w = \overline{0}))$$

4. *Primitive recursion*: let ϕ defined as follows:

   (a) $\phi(x, 0) \simeq h(x)$

   (b) $\phi(x, y + 1) \simeq g(x, y, \phi(x, y))$

   Intuitive idea. Gödel's idea was to formalize the steps of computation of $\phi(x, y) \simeq z$ as a sequence $\langle s_0, ..., s_y \rangle$, where:

   (a) $s_0 = h(x)$

   (b) $s_{i+1} = g(x, i, s_i)$

   (c) $s_y = z$

   To this goal, we need to develop functions that handle sequences. When we had primitive recursion, we could define things like the $n-$ th prime and code finite sequences by means of primes and factorization. But here we do not have primitive recursion: in fact we want to show that we can do primitive recursion. Hence Gödel used the machinery based on the so-called *Chinese remainder theorem*:

   > "Given $m_0, ..., m_k$ pairwise coprime (that is, each pair of them has no common divisors > 1), if $s_0 < m_0, ..., s_k < m_k$, then there exist a unique B such that $B < \Pi_{i \leq k} m_i$ and $Rem(B, m_i) = s_i$ (i.e. s_i is the remainder of the division of B by m_i)".

   But how to obtain a succession of $y + 1$ numbers pairwise coprime? Given $s_0, ..., s_y$ let us define $\nu = max\{s_0, ..., s_y, y+1\}$ and $A = \nu!$. Hence the numbers $1+A, 1+2A, ..., 1+(y+1)A$ are pairwise coprime (and clearly, for all $i \leq y$, $s_i < 1 + (i+1)A$). To check it, suppose that it is not true: let then p prime that divides both $1 + rA$ and $1 + r'A$; but then it will divide also their difference $(r - r')A$. Also applies in general that if $p|ab$ (with p prime), then either $p|a$, or $p|b$. Thus in our case, either $p|(r - r')$, or $p|A$. Since A is a multiple of $r - r'$ (considering that $r, r' \leq y + 1$) we have that $(r - r')|A$. Ultimately we have these alternatives:

   (a) either $p|(r - r')$, and then $p|A$,

   (b) or $p|A$.

   In both cases $p|A$. Hence $Rem(1+rA, p) = 1$ (but this is contradictory, because, by assumption $p|1 + rA$).
   Equivalent formulation of the intuitive idea. We can thus formulate the intuitive idea by saying, equivalently, that there are A, B such that:

   (a) $Rem(B, A + 1) = h(x)$

   (b) $Rem(B, A(y + 1) + 1) = z$

   (c) $\forall i < y(Rem(B, A(i + 2) + 1) = g(x, i, Rem(B, A(i + 1) + 1))$

Let us denote with (*) these conditions. Suppose these conditions (*) hold. Actually, defining $s_i = Rem(B, A(i+1)+1)$, the sequence $s_0, ..., s_y$ satisfies Gödel's intuitive idea. On the other hand, if $s_0, ..., s_y$ is a sequence that satisfies the conditions expressed by the intuitive idea and A is defined as above, then for the Chinese remainder theorem there is a number B such that each s_i can be expressed as $s_i = Rem(B, A(i+1)+1)$ in such a way that the above conditions (*) are met.
Formalization of the intuitive idea. At this point we define a formula that we denote $\beta(B, A, i, v)$, which expresses in the formal language of arithmetic the relation $(Rem(B, A \cdot (i+1)+1) = v)$, by means of a Δ_0-formula:

$$v < \overline{A} \cdot (i + \overline{1}) + \overline{1} \wedge \exists w \leq \overline{B}(\overline{B} = w \cdot (\overline{A} \cdot (i + \overline{\overline{1}}) + \overline{1}) + v)$$

Let us finally consider these three formulas:

(a) $\exists w(\beta(\overline{B}, \overline{A}, \overline{0}, w) \wedge \psi_h(x, w))$

(b) $\beta(\overline{B}, \overline{A}, u, z)$

(c) $\forall i < y \exists v \exists u(\beta(\overline{B}, \overline{A}, i, v) \wedge \beta(\overline{B}, \overline{A}, i+1, u) \wedge \psi_g(x, i, v, u))$

Note that the conjunction of these three formulas is Σ_1.
If now we call for brevity $\Theta_a(x, \overline{B}, \overline{A}), \Theta_b(\overline{B}, \overline{A}, y, z), \Theta_c(x, \overline{B}, \overline{A})$ respectively, these three formulas, finally get the desired Σ_1 formula $\psi_\phi(x, y, z)$:

$$\exists A \exists B(\Theta_a(x, B, A) \wedge \Theta_b(B, A, y, z) \wedge \Theta_c(x, B, A))$$

Note that we did not use the power-of-primes coding of sequences. The particular coding of sequences that we have shown here (the $\beta-$function) dates back from the same Gödel.

QED

As anticipated, Gödel used the notion of *representability*, restricted to recursive primitive functions (the proof of representability is almost the same), rather than definability in the model (see Odifreddi (1989-1999) 39-44).

Definition 31. *Let* T *be a theory that extends the predicate logic with identity and that contains terms* $\overline{n}$ *for all natural numbers* n*. Then, if* f *is a function, we say (for all sequence of natural numbers* $n_0, ..., n_m$*) that:*

1. f *is* weakly representable *in* T *(or* numerable*), if for some formula* ϕ *of the language of the theory* T*, we have that* $f(n_0, ..., n_m) = r$ *iff* $\mathsf{T} \vdash \phi(\overline{n_0}, ..., \overline{n_m}, \overline{r})$*.*
2. f *is* representable *in* T *(or* binumerable*), if for some formula* ϕ *of the language of* T*, we have that:*

   (a) $f(n_0, ..., n_m) = r$ *implies* $\mathsf{T} \vdash \phi(\overline{n_0}, ..., \overline{n_m}, \overline{r})$ *and*

   (b) $f(n_0, ..., n_m) \neq r$ *implies* $\mathsf{T} \vdash \neg\phi(\overline{n_0}, ..., \overline{n_m}, \overline{r})$*.*
3. f *is* strongly representable *in* T*, if moreover:*

$$\mathsf{T} \vdash \forall y \forall z(\phi(n_0, ..., n_m, y) \wedge \phi(n_0, ..., n_m, z) \rightarrow y = z)$$

The condition 3. joint to the 2.(a) is equivalent to:

$$\mathsf{T} \vdash \forall y(\phi(\overline{n_0}, ..., \overline{n_m}, y) \leftrightarrow y = \overline{f(n_0, ..., n_m)})$$

These definitions are extended to relations.

Definition 32. *Let* R *be a relation and* T *as in the previous definition:*

1. *R is* weakly representable *in* T *(or* numerable*), if for some formula ϕ of the language of* T*, we have that $R(n_0, ..., n_m)$ is true iff* $\mathsf{T} \vdash \phi(\overline{n_0}, ..., \overline{n_m})$.
2. *R is* representable *in* T *(or* binumerable*), if for some formula ϕ of the language of* T*, we have that if $R(n_0, ..., n_m)$ is true, then $\mathsf{T} \vdash \phi(\overline{n_0}, ..., \overline{n_m})$ and if $R(n_0, ..., n_m)$ is false, then* $\mathsf{T} \vdash \neg\phi(\overline{n_0}, ..., \overline{n_m})$

Let us now look at some relationships between the concepts we have introduced (see Hájek and Pudlák (1993), pp. 155-57).

Theorem 68. *The notions of representability and definability are related as follows*

1. *If R is defined in $\mathbb{N}$ by a $\phi \in \Sigma_1$, then ϕ* numerates *R in* Q.
2. *If R is Δ_1 defined in $\mathbb{N}$, then there is a $\theta \in \Sigma_1^0$ that* binumerates *R in* Q.

Proof. 1. It follows immediately from Σ_1-soundness and Σ_1-completeness of Q. Regarding 2. if $R(x_0, ..., x_n)$ is Δ_1-definable, then exists a Σ_1-formula $\exists y\phi(x_0, ..., x_n, y)$ and a Π_1-formula $\forall y\psi(x_0, ..., x_n, y)$ that define it, with $\phi, \psi \in \Delta_0$. Let now $\theta(x_0, ..., x_n)$ the formula:

$$\exists y(\phi(x_0, ..., x_n, y) \wedge \forall z \leq y\psi(x_0, ..., x_n, z))$$

Observe that:

1. if $\langle k_0, ..., k_n\rangle \in R$, then there exists an m such that $\mathbb{N} \models \phi(\overline{k_0}, ..., \overline{k_n}, \overline{m})$ and $\mathbb{N} \models \forall y\psi(\overline{k_0}, ..., \overline{k_n}, y)$. In particular, for all $s \leq m$, we have that $\mathbb{N} \models \psi(\overline{k_0}, ..., \overline{k_n}, s)$.
   It follows that Q proves $\phi(\overline{k_0}, ..., \overline{k_n}, \overline{m}) \wedge \forall z \leq \overline{m}\psi(\overline{k_0}, ..., \overline{k_n}, z)$, from which immediately $\theta(\overline{k_0}, ..., \overline{k_n})$.
2. $\langle k_0, ..., k_n\rangle \notin R$, then $\mathbb{N} \models \neg\forall y\psi(\overline{k_0}, ..., \overline{k_n}, y)$ e $\mathbb{N} \models \neg\exists y\phi(\overline{k_0}, ..., \overline{k_n}, y)$. Hence there exist an m, such that the formula:

   $$\neg\psi(\overline{k_0}, ..., \overline{k_n}, \overline{m}) \wedge \forall y \leq \overline{m}\neg\phi(\overline{k_0}, ..., \overline{k_n}, y)$$

   is true and also provable in Q.
   Reason therefore inside the theory Q and consider that if the formula $\theta(\overline{k_0}, ..., \overline{k_n})$ were true, then for some u would be true also $\phi(\overline{k_0}, ..., \overline{k_n}, u) \wedge \forall y \leq u\psi(\overline{k_0}, ..., \overline{k_n}, y)$.
   We have therefore the following alternatives:

   (a) if $u \leq \overline{m}$, then $\neg\phi(\overline{k_0}, ..., \overline{k_n}, u)$, from our assumptions, however, and for (2) we will also $\phi(\overline{k_0}, ..., \overline{k_n}, u)$ (contradiction).

   (b) $u > \overline{m}$, then $\psi(\overline{k_0}, ..., \overline{k_n}, \overline{m})$, for (2), but at the same time $\neg\psi(\overline{k_0}, ..., \overline{k_n}, \overline{m})$, by assumptions.

   Contradiction. Hence $\mathsf{Q} \vdash \neg\theta(\overline{k_0}, ..., \overline{k_n})$.

It follows that $\theta(x_0, ..., x_n)$ is the desired binumeration. QED

Corollary 15. *All recursive relations (being Δ_1-definable) are binumerable in* Q*, and all computably enumerable relations (being Σ_1-definable) are numerable in* Q.

To summarise, the following equivalences hold:

1. As for the relations, R is r.e. iff R is Σ_1-definable iff R is weakly representable (numerable). Moreover R is recursive iff R is Δ_1-definable iff R is representable (binumerable).

2. As for the functions, ϕ is a *partial* recursive function iff the graph of ϕ is Σ_1-definable iff ϕ is numerable in Q. Moreover f is a total recursive function iff the graph of f is Δ_1-definable iff f is binumerable in Q.

Axiomatizability. In the current literature, sometimes it is only required that the set of axioms or of the theorems is recursively enumerable; sometimes instead, more strictly, it is assumed that the axiomatic theories are axiomatizable in primitive recursive way. That these alternatives are ultimately equivalent, can be proved thanks to variants of a method known as "Craig's Trick". Define two theories to be deductively equivalent if they prove the same theorems. In other words, they are two different axiomatizations of a deductively closed set of formulas.

Theorem 69. *Suppose the theory* T *is computably enumerable, i.e., there is a computable function that lists the set* Thm_{T} *of its theorems; then the following are equivalent:*

1. T *is axiomatizable in primitively recursive way.*
2. Thm_{T} *is recursively enumerable.*

Proof. We proof 2. $\Rightarrow$ 1. Remember first that if a set is computably enumerable, then is the codomain of a *primitive recursive* function (see the remark on p.103). So let f primitive recursive such that $Thm_{\mathsf{T}} = f[\mathbb{N}]$. Now we apply the so-called *Craig's Trick* and replace each $f(n) = \ulcorner \alpha_n \urcorner$ with $\ulcorner \overbrace{\alpha_n \wedge ... \wedge \alpha_n}^{n-times} \urcorner$. Observe that the theory T^* axiomatized by the set so obtained is deductively equivalent to T. Moreover, being enumerable in increasing order, arguing as on p.43 it is recursive. Actually, by a similar argument, we can show that it is *primitive* recursive, since we ca check whether $\ulcorner \alpha \urcorner \in \mathsf{T}^*$ in a primitive recursive way as follows: count the number m of conjuncions in α, the compute $f(0), f(1), f(2), ..., f(m+1)$. Hence $\ulcorner \alpha \urcorner \in \mathsf{T}^*$ if it is a conjunction having code $f(i)$, for some $i \leq m+1$. 1. $\Rightarrow$ 2. we already know that Thm_{T} is creative. More constructively we will see that in the relation:

$$\exists d[d \text{ is a proof of } \phi \text{ from } \Gamma]$$

the part in brackets is computable, when Γ is computable. Note that we have defined Thm_{T} by means of a Σ_1 formula. QED

4.2. Arithmetization of metamathematics

We need to encode metamathematical concepts: certain assertions about formulas will be converted into assertions about natural numbers and thus expressed in the formal language, and therefore we will express facts *about formulas* by expressing facts *about numbers. Arithmetizing* a certain discrete domain means encode objects and notions of that domain using natural numbers; in particular, symbols, formulas (thought of as chains of symbols) and statements about formulas of a formal language can be converted to natural numbers and statements about the natural numbers using a numerical coding.

Suppose we have assigned a number to each logical constant, e.g.:

$\forall$	$\exists$	$\wedge$	$\vee$	$\neg$	$\rightarrow$	$=$	(	)
$\langle 0,0\rangle$	$\langle 0,1\rangle$	$\langle 0,2\rangle$	$\langle 0,3\rangle$	$\langle 0,4\rangle$	$\langle 0,5\rangle$	$\langle 0,6\rangle$	$\langle 0,7\rangle$	$\langle 0,8\rangle$

1. variables x_i are coded as pairs $\langle 1, i\rangle$ (Possibly i can be written in unary or binary notation).
2. a constant c_i will be coded by $\langle 2, i\rangle$;
3. an n-ary function symbol f_j will be coded as $\langle 3, j, n\rangle$;
4. an n-ary relational symbol P_j will be coded by $\langle 4, j, n\rangle$

If $s_0, ..., s_n$ is a sequence of symbols, and c_i is the code of the symbol s_i, its Gödel number is the coded sequence:

$$\langle c_0, ..., c_n\rangle = p_0^{n+1} \cdot p_1^{c_0} \cdot ... \cdot p_n^{c_n}$$

1. a generic composite term $f_j(t_0, ..., t_m)$ will be coded as a sequence:

$$\ulcorner f_j(t_0, ..., t_m)\urcorner = \langle \ulcorner f_j \urcorner, \ulcorner (\urcorner, \ulcorner t_0 \urcorner, ..., \ulcorner t_m \urcorner, \ulcorner) \urcorner\rangle$$

2. A generic atomic formula $P_j(t_0, ..., t_m)$ will be coded as a sequence:

$$\ulcorner P_j(t_0, ..., t_m)\urcorner = \langle \ulcorner P_j \urcorner, \ulcorner (\urcorner, \ulcorner t_0 \urcorner, ..., \ulcorner t_m \urcorner, \ulcorner) \urcorner \rangle$$

The non atomic formulas will be finally codified in this way:

(a) $\ulcorner(\neg\phi)\urcorner = \langle \ulcorner(\urcorner, \ulcorner\neg\urcorner, \ulcorner\phi\urcorner, \ulcorner)\urcorner\rangle$

(b) $\ulcorner(\alpha \vee \beta)\urcorner = \langle \ulcorner(\urcorner, \ulcorner\alpha\urcorner, \ulcorner\vee\urcorner, \ulcorner\beta\urcorner, \ulcorner)\urcorner\rangle$

(c) $\ulcorner(\alpha \wedge \beta)\urcorner = \langle \ulcorner(\urcorner, \ulcorner\alpha\urcorner, \ulcorner\wedge\urcorner, \ulcorner\beta\urcorner\ulcorner)\urcorner\rangle$

(d) $\ulcorner(\alpha \leftrightarrow \beta)\urcorner = \langle \ulcorner(\urcorner, \ulcorner\alpha\urcorner, \ulcorner\leftrightarrow\urcorner, \ulcorner\beta\urcorner, \ulcorner)\urcorner\rangle$

(e) $\ulcorner(\alpha \rightarrow \beta)\urcorner = \langle \ulcorner(\urcorner, \ulcorner\alpha\urcorner, \ulcorner\rightarrow\urcorner, \ulcorner\beta\urcorner, \ulcorner)\urcorner\rangle$

(f) $\ulcorner(\forall x_j \beta)\urcorner = \langle \ulcorner(\urcorner, \ulcorner\forall\urcorner, \langle 0, j\rangle, \ulcorner\beta\urcorner . \ulcorner)\urcorner\rangle$

(g) $\ulcorner(\exists x_j \beta)\urcorner = \langle \ulcorner(\urcorner, \ulcorner\exists\urcorner, \langle 0, j\rangle, \ulcorner\beta\urcorner, \ulcorner)\urcorner\rangle$

The idea goes back to Leibniz. There are many sophisticated ways to accomplish this, which in some cases take into account the complexity, but the essential point is that it is always possible to pass, purely mechanically, from an expression to its code number, and from a number to the corresponding expression. In turn, these coded expressions can be translated into the language of the same theory, if is sufficiently expressive: this is therefore a method which enables the theory *to speak indirectly of itself*, stratagem that allows us to formalize an argument based on *self-reference*.

The relations "x is a variable", "x is a constant", "x is an $n-$ary function symbol", "x is an $n-$ary predicative symbol", are clearly primitive recursive. For example "x is a variable" is defined as $Var(x) = \exists y < x(x = \langle 1, y\rangle)$ (analogously for the other concepts).

Lemma 26. *The relations $Term(x)$ and $Form(x)$, respectively "x is a variable", "x is a constant", "x is a term" and "x is a formula", are primitive recursive.*

Proof. To show that the predicate $Term(x)$= "x is a term" is primitive recursive, we build a 'constructional history' for a term, or a term-sequence. Let $Termseq(n)$ encodes a term sequence, i.e. a sequence of expressions $t_0, t_1, ..., t_n$ such that each expression t_i in the sequence either is a constant; or is a variable; or else is built by using an $m - place$ function symbol from m terms occurring prior to place i:

$$Termseq(n) = Seq(n) \wedge \forall k \leq lh(n)[\Psi_0 \vee \Psi_1 \vee \Psi_2 \vee \Psi_3]$$

where:

1. $\Psi_0 = (n)_k = \ulcorner\overline{0}\urcorner$
2. $\Psi_1 = \exists j < k((n)_k = \langle \ulcorner S\urcorner, (n)_j\rangle)$
3. $\Psi_2 = \exists j < k \exists i < k((n)_k = \langle \ulcorner +\urcorner, (n)_i, (n)_j\rangle$
4. $\Psi_3 = \exists j < k \exists i < k((n)_k = \langle \ulcorner \cdot\urcorner, (n)_i, (n)_j\rangle$

Hence $Term(x)$ can be defined as $\exists y(Termseq(y) \wedge ((y)_{lh(y)} = x)$. But we need to find a bound to the existential quantifier. Notice that the last element x of the sequence is the bigger and this sequence y has the form $p_0^n \cdot p_1^{x_1} \cdot ... \cdot p_n^{x_n}$ where $x_n = x$. Hence a bound could be $y < (p_x!)^x$.

Now $Form(z)$ can be defined repeating the same strategy we used in defining Term where $Formseq(x)$ is a sequence such that for all $i \leq lh(x)$, either:

1. $(x)_i = \langle \ulcorner=\urcorner, y, z\rangle \wedge Term(y) \wedge Term(z)$, or
2. $(x)_i = \langle \ulcorner\wedge\urcorner, y, z\rangle \wedge Form(y) \wedge Form(z)$, or
3. (analogous conditions by replacing the conjunction with the other connectives), or

4. $(x)_i = \langle \ulcorner\forall\urcorner, y, z\rangle \wedge Var(y) \wedge Form(z)$, or
5. (analogous condition for the existential quantifier)".

where $y, z < x$, and the last element is z.

QED

Along the same lines, it is shown that the following are primitive recursive :

1. $ClTerm(x)$, "x is a closed term".
2. $FTVar(x, y)$, "y is a term that contains free occurrences of the variable with Godel number x".
3. $FFVar(x, y)$, "x is a free variable in the formula y".
4. $FreeF(x, y, z)$, "x is a term free for variable y in the formula z".
5. $Sent(x)$," x is the Gödel number of a sentence is primitive recursive."

If we consider axiomatic systems, the following are primitive recursive (for the sake of readability we will abbreviate $\langle\ulcorner(\urcorner, x, \ulcorner\rightarrow\urcorner, y, \ulcorner)\urcorner\rangle$ with $x\dot{\rightarrow}y$):

1. $Equax(x)$, "x encoding an axiom of equality".
2. $PropAx(x)$, "x is a propositional axiom".
3. $QAx(x)$, "x is a quantifiers axiom". Recall that the quantifier axioms are $\phi(t) \rightarrow \exists x\phi$ and $\forall x\phi \rightarrow \phi(t)$ (t free for x in ϕ). To say this we must say that there are $y, z, w \leq x$ such that the conjunction of the following formulas holds:

   (a) $Var(y) \wedge Form(z) \wedge Term(w) \wedge FreeF(w, y, z)$

   (b)
   $$(x = \langle\ulcorner(\urcorner, \ulcorner\forall\urcorner, y, z, \ulcorner)\urcorner\rangle\dot{\rightarrow}Subst(z, y, w))\vee$$
   $$\vee(x = Subst(z, y, w)\dot{\rightarrow}\langle\ulcorner(\urcorner, \ulcorner\exists\urcorner, y, z, \ulcorner)\urcorner\rangle)$$

4. $Logax(x)$ iff "x code a logical axiom" i.e. the disjunction of $PropAx(x)$ and $QAx(x)$.
5. $MP(x, y, z)$ iff "y is obtained from z and x by Modus Ponens" as $z = x\dot{\rightarrow}y$.
6. Recall the rule of existential generalization: "from $\phi \rightarrow \psi$ conclude $\exists u\phi \rightarrow \psi$, assuming u not free in ψ". Now "x is obtained from y by existential generalization" can be expressed in primitive recursive way:

   $$\begin{aligned}&\exists v < x\exists w < x\exists z < x(Var(v) \wedge Form(w) \wedge Form(z)\wedge \\ &\wedge \neg FFvar(v, z) \wedge y = w\dot{\rightarrow}z \wedge x = \langle\ulcorner(\urcorner, \ulcorner\exists\urcorner, v, w, \ulcorner)\urcorner\rangle)\dot{\rightarrow}z)\end{aligned} \tag{2}$$

   We could do the same for the universal generalization: "from $\phi \rightarrow \psi$ conclude $\phi \rightarrow \forall u\psi$, where u not free in ϕ", and so clearly we could then express $Gen(y, x)$ as " x is obtained from y by generalization (existential or universal)".

In the above, we also used this primitive recursive *substitution function*:

$$Subst(\ulcorner\phi\urcorner, i, \ulcorner t\urcorner) = \ulcorner\phi[t/x_i]\urcorner$$

whose effect is to take (Gödel number) of a formula $\phi(x_i)$ and replace in it the term t, in place of all occurrences of the variable x_i. This can be obtained by primitive recursion on the course of values:

1. Terms:

   (a)

$$Subst(\ulcorner x_j \urcorner, i, y) = \begin{cases} y & \text{if } i = j \\ \ulcorner x_i \urcorner & \text{if } i \neq j \end{cases}$$

   (b) $Subst(\ulcorner f_i(t_0, ..., t_n) \urcorner, j, y) = \langle \ulcorner f_i \urcorner, Subst(\ulcorner t_0 \urcorner, i, y), ..., Subst(\ulcorner t_n \urcorner, i, y) \rangle$

2. Formulas:

   (a) $Subst(\ulcorner R_i(t_0, ..., t_n) \urcorner, j, y) = \langle \ulcorner R_i \urcorner, Subst(\ulcorner t_0 \urcorner, i, y), ..., Subst(\ulcorner t_n \urcorner, i, y) \rangle$

   (b) $Subst(\ulcorner \neg\phi \urcorner, j, y) = \langle \ulcorner \neg \urcorner, Subst(\ulcorner \phi \urcorner, j, y) \rangle$

   (c) $Subst(\ulcorner \phi \wedge \psi \urcorner, j, y) = \langle \ulcorner \wedge \urcorner, Subst(\ulcorner \phi \urcorner, j, y), Subst(\ulcorner \psi \urcorner, j, y) \rangle$ (analogously for $\vee, \rightarrow$)

   (d)

$$Subst(\ulcorner \forall x_i \phi \urcorner, j, y) = \begin{cases} \langle \ulcorner \forall \urcorner, \ulcorner x_i \urcorner, Subst(\ulcorner \phi \urcorner, j, y) \rangle & \text{if } i \neq j \\ \ulcorner \forall x_i \phi \urcorner & \text{if } i = j \end{cases}$$

   (analogously for $\exists$)

   (e) $Subst(x, j, y) = 0$ in all other cases.

The provability predicate

For our purposes we need in particular a primitive recursive relation $\mathsf{Prf}_\mathsf{T}(x, y)$ whose intende meaning is "x code a correct proof of y in the theory T". Let's say that a theory is *presented in primitive recursive way* if the predicate Ax_T that defines the proper axioms of the theory T is in turn primitive recursive; in familiar theories of arithmetic that we have previously here quoted, the Ax_T it is. The complexity of the provability predicate $\mathsf{Prf}_\mathsf{T}(x, y)$ depends essentially on that of the formula Ax_T. To fix the ideas, let $\mathsf{T} = \mathsf{PA}$, then $Ax_\mathsf{PA}(y)$, a standard definition of proper axioms has the form $a_0 \vee ... \vee a_7 \vee Ind(y)$, where $a_0, ..., a_7$ are the codes of the eight axioms of *Peano Arithmetic* different from the induction, and $Ind(y)$ is true iff y codes an instance of the induction principle. To formalize the predicate $Ind(y)$ the idea is to say: "there are $x, z < y$ such that $Form(x)$ and $\exists i < z(z = \langle 1, i \rangle)$ and if $Subst(x, i, \ulcorner \overline{0} \urcorner)$ and for all v, if $Subst(x, i, \ulcorner \overline{v} \urcorner)$, then $Subst(x, i, \ulcorner \overline{S(v)} \urcorner)$, then for all v, $Subst(x, i, \ulcorner \overline{v} \urcorner)$". The provability predicate $\mathsf{Prf}_\mathsf{T}(x, y)$, "$x$ codes a proof in T of y", is now defined as:

$$\begin{gathered} Seq(x) \wedge ((x)_{ln(x)} = y) \wedge \\ \wedge \forall i \leq lh(x)(Logax((x)_i) \vee Equax((x)_i) \vee Ax_\mathsf{T}((x)_i) \vee \\ \vee \exists h, j < iMP((x)_h, (x)_j, (x)_i) \vee \exists j < iGen((x)_j, (x)_i)) \end{gathered} \tag{3}$$

where, according to our coding of sequences, $ln(x)$ denotes the length of the sequence x and is defined as $(x)_0$. In what follows, we will denote by Prf_T the formula in the language of Peano Arithmetic that represents the primitive recursive predicate of provability Prf_T.

Remark 4. *We emphasise once again that we have defined the predicate of provability for Hilbert-style axiomatic systems, where proofs are sequences of formulas, solely for the sake of simplicity of exposition. A similar definition is possible for Gentzen-style systems, where derivations are labeled trees, although it is more laborious (see for instance Van Dalen (2013) 248-51 for Natural Deduction, or Buss (1986) 130-34 and Girard (1987) for Sequent Calculus).*

4.3. Syntactic proofs of Gödel's theorems

> we can readily see that the proof just given is constructive; that is...proved in an intuitionistically unobjectionable manner... (K. Gödel, 1931)

That the 1931 proof of the first theorem is *syntactic* and *constructive*. This means that it doesn't appeal to truth and that we concretely *give examples* of sentences that are independent (neither provable nor refutable). We start with the central result from which we derive the first Gödel theorem.

Theorem 70. (Fixed point theorem) *Let* T *a consistent extension of* Q *in the same language; then for all formulas* $\psi(x)$*, exists another formula* ϕ*, such that:*

$$\mathsf{T} \vdash \phi \leftrightarrow \psi(\overline{\ulcorner \phi \urcorner})$$

We say that ϕ *costitutes a* fixed point *of* ψ.

Proof. Let $Num(x)$ the (easily proved promitive recursive) function whose effect is to take a number n and return Gödel's number of the numeral of that number $\ulcorner \overline{n} \urcorner$ and let therefore $Sub(\ulcorner \phi(x_i) \urcorner, n) = Subst(\ulcorner \phi(x_i) \urcorner, i, Num(n)) = \ulcorner \phi(\overline{n}) \urcorner)$. First we remind the reader that according to what we said around definability and representability of functions if f is recursive we have:

$$\mathsf{Q} \vdash \alpha_f(\overline{m}, y) \leftrightarrow (y = \overline{f(m)})$$

In particular, if $f(x) = Sub(x, x)$, then exists a formula $\alpha(x, y)$ such that $\mathsf{T} \vdash \alpha(\overline{n}, v) \leftrightarrow (v = \overline{Sub(n, n)})$. Let therefore $\chi(x)$ the formula $\exists v(\alpha(x, v) \wedge \psi(v))$; then suppose that $\ulcorner \chi(x) \urcorner = m$ and lastly let ϕ the formula $\chi(\overline{m})$, i.e.the formula χ "applied" to itself.

We see that the following holds in T:

$$\begin{aligned}
\phi &\longleftrightarrow \chi(\overline{m}) \\
&\longleftrightarrow \exists v(\alpha(\overline{m}, v) \wedge \psi(v)) \\
&\longleftrightarrow \exists v(v = \overline{Sub(\ulcorner \chi(x) \urcorner, \ulcorner \chi(x) \urcorner)} \wedge \psi(v)) \\
&\longleftrightarrow \exists v(v = \overline{\ulcorner \phi \urcorner} \wedge \psi(v)) \\
&\longleftrightarrow \psi(\overline{\ulcorner \phi \urcorner})
\end{aligned} \tag{4}$$

QED

A first remarkable application of this result, is the Tarski theorem

Theorem 71. (Tarski's undefinability theorem 1933) *A sufficiently strong consistent theory can not express its truth.*

Proof. To make the proof more accessible, we give a semantic version and we show that there is no definition of $Th(\mathbb{N})$ in the standard model. So we consider a semantic version of the fixed point theorem, for which we shall have $\mathbb{N} \models \phi$ iff $\mathbb{N} \models \psi(\overline{\ulcorner \phi \urcorner})$. Suppose by contradiction that $Th(\mathbb{N})$ is definable, i.e. there exist a "definition of truth", namely a a formula $True(x)$ such that $\mathbb{N} \models \phi$ iff $\ulcorner \phi \urcorner \in Th(\mathbb{N})$ iff $\mathbb{N} \models True(\overline{\ulcorner \phi \urcorner})$. Take a fixed point of $\neg True(x)$, let for instance τ this fixed point; then we have: $\mathbb{N} \models \neg True(\overline{\ulcorner \tau \urcorner})$ iff $\mathbb{N} \models \tau$ iff $\mathbb{N} \models True(\overline{\ulcorner \tau \urcorner})$, a contradiction. QED

A language that contains its own truth predicate and names for all its sentences, is called by Tarski *semantically closed* (e.g. the natural languages). The notion of truth cannot be defined in these languages. However many scientific languages are not semantically closed and can be placed in a hierarchy, where the truth for an obiect language can be define in a higher meta-language. In particular there is a *second order* truth-definition for first order sentences over $\mathbb{N}$. That is, $Th(\mathbb{N})$ is an analytical set. As remarked in the introductory chapter, there are also *partial* truth-predicates for all fixed level of arithmetical hierarchy and this can be proved already in the subteory of PA with induction restricted to Σ_1 formulas.

The first incompleteness theorem receives today several presentations which nevertheless reflect the same general structure (see Hájek and Pudlák (1993), ch.III):

1. It is required that the formal theory of arithmetic T is an extension of Q, which thus proves the *Fixed point theorem*, and also that the set of its theorems is recursively enumerable, and therefore have a Σ_1-definition $P(x)$.

2. Then we take a fixed point ν of the formula $\neg P(x)$, i.e. $\nu \longleftrightarrow \neg P(\overline{\ulcorner \nu \urcorner})$ where the fixed point ν is just Gödel's sentence, that will be proved undecidable.

3. Hence we prove that:

   (a) If T is consistent, then $T \nvdash \nu$

   (b) Under certain conditions (e.g. 1-consistency) $T \nvdash \neg\nu$.

1. In fact, Gödel did not consider a generic enumeration of theorems., but one based on a particular formula, namely the Σ_1 formula that enumerates the theorems of T based on the formula that represents the provability predicate, i.e. take $P(x) = \exists y \mathsf{Prf_T}(y, x)$. Thus the theory T shows the existence of a fixed point ν for its *negation*:

$$\nu \longleftrightarrow \neg \exists y \mathsf{Prf_T}(y, \overline{\ulcorner \nu \urcorner})$$

The formula $\exists y \mathsf{Prf_T}(x, y)$ is usually abbreviated as $\mathsf{Pr_T}(\overline{\ulcorner \nu \urcorner})$.

Note that ν says of itselfs of not being provable: hence, if it is unprovable it will be true. Note also the resemblance to the paradox of the liar and so with the Tarski theorem of undefinability of truth, where in place of provability (definable) appears the truth (undefinable). But what are these " additional conditions "? Alternatively, one of the following is usually placed:

1. $\omega-$consistency. A theory is ω-consistent, iff it is not the case that $\mathsf{T} \vdash \exists x \neg\phi(x)$ and $\mathsf{T} \vdash \phi(\overline{0}), \mathsf{T} \vdash \phi(\overline{1}), \mathsf{T} \vdash \phi(\overline{2}), ...etc.$ for all natural numbers. This is the original approach of K. Gödel's paper of 1931.

2. Σ_1-*soundness* (or 1-consistency). Kreisel observed that the assumption of ω-consistency was unnecessarily strong and can be replaced by 1-consistency, namely ω-consistency restricted to Σ_1 sentences. This property is equivalent to Σ_1-soundness, namely the property according to which if a Σ_1 sentence is provable, then it is true. Note that this property implies consistency. It is not difficult to check that, if we denote with $Th_{\Pi_1}(\mathbb{N})$ the set of true Π_1 sentences, then we have that T is Σ_1-sound, iff $T + Th_{\Pi_1}(\mathbb{N})$ is consistent.

Let's see some relations among these concepts. If a theory is $\omega-$ consistent, then is Σ_1-sound: indeed suppose that $\mathbb{N} \not\models \exists x\theta$, where $\exists x\theta \in \Sigma_1$; hence $\mathbb{N} \models \neg\exists x\theta$ and therefore $\mathbb{N} \models \neg\theta(\overline{n})$, for all numbers n. But $\theta \in \Delta_0$ and therefore we will have also $\mathsf{T} \vdash \neg\theta(\overline{n})$, for all numbers n and by $\omega-$ consistency $T \nvdash \exists x\theta$. If a theory T is ω-consistent, then is also consistent: note indeed that $\mathsf{T} \vdash x = x$ and therefore $\mathsf{T} \vdash \overline{n} = \overline{n}$ for all n; hence $T \nvdash \exists x(x \neq x)$ by ω-consistency, and therefore T is consistent (i.e. there is at least an unprovable sentence). The opposite direction does not hold and therefore ω-consistency is a notion strictly stronger than consistency. Let us consider for example $\mathsf{T} = \mathsf{PA} + \mathsf{Pr_{PA}}(\overline{\ulcorner \overline{1} = \overline{0} \urcorner})$. Since $\mathsf{PA} \nvdash \neg\mathsf{Pr_{PA}}(\overline{\ulcorner \overline{1} = \overline{0} \urcorner})$, we have that T is consistent. Moreover for all n, the relation $\neg\mathsf{Prf_{PA}}(n, \overline{\ulcorner \overline{1} = \overline{0} \urcorner})$ is true. Hence, by binumerablility, $\mathsf{PA} \vdash \neg\mathsf{Prf_{PA}}(\overline{n}, \overline{\ulcorner \overline{1} = \overline{0} \urcorner})$, from which $\mathsf{T} \vdash \neg\mathsf{Prf_{PA}}(\overline{n}, \overline{\ulcorner \overline{1} = \overline{0} \urcorner})$ and at the same time $\mathsf{T} \vdash \exists y \mathsf{Prf_{PA}}(y, \overline{\ulcorner \overline{1} = \overline{0} \urcorner})$.

Gödel's first incompleteness theorem. Let's see how we can get the first theorem, in a version closer to the original due to Gödel, through the concepts of ω-consistency and binumerability, and the provability predicate. Let ν a fixed point of $\neg\mathsf{Pr_T}$, where T is an extension of Q. That is, the following is provable it:

$$\nu \leftrightarrow \neg\mathsf{Pr_T}(\overline{\ulcorner \nu \urcorner})$$

We show that:

(a) under the hypothesis of *consistency*, ν is not provable in T, e

(b) under the hypothesis $\omega - consistency$, is not provable neither in $\neg\nu$.

Proof. Let's look at the two cases:

(a) Let us suppose that $\mathsf{T} \vdash \nu$; then a number n codes a proof of ν in T. But the primitive recursive relation "n codes a proof of ν in T", as we have seen, is binumerable in T, i.e. we have $\mathsf{T} \vdash \mathsf{Prf_T}(\overline{n}, \overline{\ulcorner\nu\urcorner})$. Hence $\mathsf{T} \vdash \exists x \mathsf{Prf_T}(x, \overline{\ulcorner\nu\urcorner})$, namely $\mathsf{T} \vdash \neg\nu$, from the definition of ν, against consistency.

(b) Suppose, therefore, to have demonstrated (a) in the way that we said; if T is ω-consistent, then is also consistent and therefore $\mathsf{T} \nvdash \nu$; as a consequence, for no natural number n we will have that it codes a proof of ν in T, namely, the primitive recursive relation "n codes a proof of ν" does not hold for any natural number; it follows that $\mathsf{T} \vdash \neg\mathsf{Prf_T}(\overline{n}, \overline{\ulcorner\nu\urcorner})$, for all number n. Lastly, from ω-consistency it follows $\mathsf{T} \nvdash \exists y \mathsf{Prf_T}(y, \overline{\ulcorner\nu\urcorner})$ and therefore $T \nvdash \neg\nu$.

QED

Rosser's version of the first incompleteness theorem. The assumption of $\omega-$consistency may seem too restrictive, in view of generalizations of the method to generic consistent theories that extend Q: we mentioned an example of a theory that is consistent, though not $\omega-$consistent. Similarly if we take the indipendent sentence ν of Gödel's theorem, being of complexity Π_1, its negation will be Σ_1; then $\mathsf{Q} + \neg\nu$ will be consistent (since ν is not derivabile from Q), but is not Σ_1-sound: indeed clearly $\mathsf{Q} + \neg\nu \vdash \neg\nu$, but $\neg\nu$ is *false* (being Gödel sentence ν true). A result shown by J.B. Rosser in 1936, allows us to use only the hypothesis of simple *consistency*, for both (3.a) that for (3.b). It makes use of the technique of the so-called *witness comparison* to produce a provability predicate rather particular. Note that the following proof uses only elementary means already available in Robinson's Arithmetic.

Theorem 72. (Rosser 1936) *Let* $\mathsf{T} \supseteq \mathsf{Q}$ *computably enumerable consistent; then exists un sentence* β*, such that (a)* $\mathsf{T} \nvdash \beta$ *e (b)* $\mathsf{T} \nvdash \neg\beta$.

Proof. Check this for Q. Let $\mathsf{Pr^R_Q}(\overline{\ulcorner\alpha\urcorner})$ be the following sentence:

$$\exists x(\mathsf{Prf_Q}(x, \overline{\ulcorner\alpha\urcorner}) \wedge \forall y < x \neg\mathsf{Prf_Q}(y, \overline{\ulcorner\neg\alpha\urcorner}))$$

and consider the fixed point $\beta \leftrightarrow \neg\mathsf{Pr^R_Q}(\overline{\ulcorner\beta\urcorner})$.

(a) Suppose that $\mathsf{Q} \vdash \beta$ and let a a number that witness this, coding a proof of β; from binumerability we have $\mathsf{Q} \vdash \mathsf{Prf_Q}(\overline{a}, \overline{\ulcorner\beta\urcorner})$ and $\mathsf{Q} \vdash \neg\mathsf{Prf_Q}(\overline{n}, \overline{\ulcorner\neg\beta\urcorner})$ for all n. Using the principle, provable in Q, according to which $\forall x \leq \overline{a}(\bigvee_{n \leq a} x = \overline{n})$ we obtain:

$$\exists x(\mathsf{Prf_Q}(x, \overline{\ulcorner\beta\urcorner}) \wedge \forall y < x \neg\mathsf{Prf_Q}(y, \overline{\ulcorner\neg\beta\urcorner}))$$

namely: $\mathsf{Q} \vdash \neg\beta$, against consistency.

Indeed, thanks to the identity axioms:

$$\neg\mathsf{Prf_Q}(\overline{n}, \overline{\ulcorner\neg\beta\urcorner})) \rightarrow (x = \overline{n} \rightarrow \neg\mathsf{Prf_Q}(x, \overline{\ulcorner\neg\beta\urcorner}))$$

and since we have $\neg\mathsf{Prf_Q}(\overline{n}, \overline{\ulcorner\neg\beta\urcorner}))$ for all n, by "modus ponens" we get, for all n, $x = \overline{n} \rightarrow \neg\mathsf{Prf_Q}(x, \overline{\ulcorner\neg\beta\urcorner})$.

Now we use the tautology

$$((A \rightarrow C) \wedge (B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$$

From the conjunction $\bigwedge_{n \leq a}(x = \overline{n} \rightarrow \neg\mathsf{Prf_Q}(x, \overline{\ulcorner\neg\beta\urcorner})$ we get therefore the formula $((\bigvee_{n \leq a} x = \overline{n}) \rightarrow \neg\mathsf{Prf_Q}(x, \overline{\ulcorner\neg\beta\urcorner})$ and from the Q theorem $\forall x(x \leq \overline{a} \rightarrow \bigvee_{n \leq a} x = \overline{n})$ by modus ponens from these two formulas we obtain $x \leq \overline{a} \rightarrow \neg\mathsf{Prf_Q}(x, \overline{\ulcorner\neg\beta\urcorner})$, namely $\forall x \leq \overline{a}(\neg\mathsf{Prf_Q}(x, \overline{\ulcorner\neg\beta\urcorner})$, from which, having already $\mathsf{Prf_Q}(\overline{a}, \overline{\ulcorner\beta\urcorner})$, it follows:

$$\exists x(\mathsf{Prf_Q}(x, \overline{\ulcorner\beta\urcorner}) \wedge \forall y < x \neg\mathsf{Prf_Q}(y, \overline{\ulcorner\neg\beta\urcorner}))$$

(b) Suppose that $\mathsf{Q} \vdash \neg\beta$ and let a a number that witness this, coding a proof of $\neg\beta$; it follows that no b and in particular no $b \leq a$ codes a proof of β (consistency). This relation is binumerable, hence we have:

(i) $\mathsf{Q} \vdash \mathsf{Prf}_{\mathsf{Q}}(\overline{a}, \overline{\ulcorner\neg\beta\urcorner})$

(ii) $\mathsf{Q} \vdash \neg\mathsf{Prf}_{\mathsf{Q}}(\overline{b}, \overline{\ulcorner\beta\urcorner})$, for all number $b \leq a$.

However, by using, as in the previous case, the identity axioms and the principle:

$$\forall x(x \leq \overline{a} \rightarrow \bigvee_{n\leq a} x = \overline{n})$$

from (ii) we get $\forall y(y \leq \overline{a} \rightarrow \neg\mathsf{Prf}_{\mathsf{Q}}(y, \overline{\ulcorner\beta\urcorner})$ that (again by axioms of Q) is equivalent to $\forall y(\mathsf{Prf}_{\mathsf{Q}}(y, \overline{\ulcorner\beta\urcorner}) \rightarrow \overline{a} < y)$, while from (i) follows $\forall y(\overline{a} < y \rightarrow \exists z < y\mathsf{Prf}_{\mathsf{Q}}(z, \overline{\ulcorner\neg\beta\urcorner})$. From these last two formulas obtained thereby follows:

$$\forall x(\mathsf{Prf}_{\mathsf{Q}}(x, \overline{\ulcorner\beta\urcorner}) \rightarrow \exists y < x\neg\mathsf{Prf}_{\mathsf{Q}}(y, \overline{\ulcorner\neg\beta\urcorner}))$$

namely $\mathsf{Q} \vdash \beta$ (against consistency).

QED

4.4. The limit of incompleteness

But how much mathematical information is actually *needed* for obtaining the first incompleteness theorem? Is Robinson's Q the best framework for explaining incompleteness and undecidability? Since we want to consider theories in different languages, we need a method to compare them and for this reason we introduce the notion (due to Tarski) of *relative interpretability* between theories, as a measure of their strength, so that Gödel's theorem, instead of speaking of consistent extensions of Q, can be rephrased as follows:

Each consistent axiomatizable theory S that *interprets* Q is incomplete.

Definition 33. *An interpretation of a theory* S *in a theory consists of a pair* $\langle\delta(x), \tau\rangle$*:*

1. *The formula* $\delta(x)$ *of the language of* T *is called domain of the intepretation (the objects of* S *from the point of view of* T*) and* τ *is a computable function from the language of* S *to the language of* T*.*
2. *There are definitions in* T *of all symbols of* S*. The translation* τ *sends each* $n-ary$ *relational symbol* R *in a formula* ψ_R *with the same ariety; maps each* $n-ary$ *functional symbol* f *in a formula* ψ_f *of ariety* $n-1$ *and each constats (i.e.* $0-ary$ *function) in a formula* $\psi_c(y)$*.*
3. *The translation extends to terms and atomic formulas. Let us denote* $(t^{\tau,w}) = w$ *is the value of* t *according to* τ*:*
   (a) *variables:* $(x^{\tau,w}) = (w = x)$
   (b) *Constants:* $(c^{\tau,w}) = \psi_c(w)$
   (c) *Functions.* $(f(t_0, ..., t_n)^{\tau,w}) = \exists w_0, ..., \exists w_n(\bigwedge_i \delta(w_i) \wedge \bigwedge_i(t_i^{\tau,w_i}) \wedge \psi_f(w_0, ..., w_n, w)$
   (d) *Atomic formulas.* $R(t_0, ..., t_n)^{\tau} = \exists v_0, ..., \exists v_n(\bigwedge_i \delta(v_i) \wedge \bigwedge_i(t_i^{\tau,v_i}) \wedge \psi_R(v_0, ..., v_n))$
4. *This translation* τ *commutes with connectives and relativises to* $\delta(x)$ *the quantifiers:*
   (a) $(\forall x\theta)^{\tau} = \forall x(\delta(x) \rightarrow \theta^{\tau})$
   (b) $(\exists x\theta)^{\tau} = \exists x(\delta(x) \wedge \theta^{\tau})$
5. S *is relatively intepretable in* T *if there exists such a pair* $\langle\delta(x), \tau\rangle$ *and:*

(a) $\mathsf{T} \vdash \exists x \delta(x)$

(b) *If f is a symbol of s, then* T *proves:*

$$\bigwedge_i \delta(x_i) \rightarrow \exists! y(\delta(y) \wedge \psi_f(x_0, ..., x_n, y))$$

(c) *In particular, for constants c,* T *proves* $\exists! y(\delta(y) \wedge \psi_c(y))$

(d) *It is asked that the translations of all axioms of S are theorems of* T.

(e) *The interpretation is transitive and reflexive.*

Theorem 73. *The following hold:*

1. *If the theory* S *is intepretable in a theory* T*, then the consistency of* T *implies the consistency of* S.
2. *Moreover, if* S *is essentially undecidable, then* T *is essentially undecidable too.*

Proof. (See e.g. Murawski (1999), pp. 250-260) QED

The notion of interpretability can therefore be used as a means to measure strength of axiomatic theories: if T and S are mutually interpretable, then we can think that they represent the same expressive and deductive strength. A theory has a minimal degree of interpretation if no theory is strictly intepretable in it. It is an open question whether a theory S exists that is *minimal* in this sense, i.e. no theory is stricty interpretable in it, and Gödel result holds. We know that the other Robinson's theory, named R, is essentially incomplete and essentially undecidable too: can we find a theory S strictly intepretable in it and such that these results hold? Actually there are several of these theories: Jerabék, for example, has found an essentially undecidable and essentially incomplete theory that is unable to interpret R.

Considering theories formulated in different languages, for instance, the following set theory with only these two axioms:

1. $\exists x \forall y \neg(y \in x)$
2. $\forall x \forall y \exists z \forall v(v \in z \leftrightarrow (v = x \vee v = y))$

denoted AST (Adjunctive Set Theory), has the same strength as Q, since the two theories are mutually interpretable (see Montagna and Mancini (1994) for an in-depth study of extremely weak set theories and Švejdar (2007) for the mutual interpretability between Q and the theory of string concatenation). About the somewhat neglected "little sister" R we can actually say more:

Theorem 74. (Visser 2009) T *is locally finitely satisfable (i.e. any finite subtheory has a finite model) iff it is intepretable in* R.

Proof. (see Visser (2009) and here on p.193.) QED

Intepretability logic was deeply investigated in a multi-modal logics framework, with an additional binary modality $\phi \rhd_{\mathsf{T}} \psi$, whose intended meaning is that there is a relative interpretation of $\mathsf{T} + \psi$ in $\mathsf{T} + \phi$. It was the subject of intense studies, started partly by the Dutch school De Jongh and Veltman (1990), Visser (1990), partly by the Italian school Montagna (1987), Berarducci (1990), partly by the Czech school Švejdar (1983) and Russian school Shavrukov (1988).

4.5. A complete and decidable theory

We have already mentioned in the introduction some historical examples of decidable theories, such as the theory of real ordered fields. The example we are considering here instead concerns a theory closer to those analyzed in this chapter, namely *the first-order theory of the addition* of the natural numbers. This result was proven independently by M. Presburger in 1929 and Thoralf Skolem in 1930 by the method of quantifier-elimination, first introduced in 1919 by Skolem in proving the completeness and decidability of the fIrst-order theory of a special class of boolean algebras. The full details were first published by Paul Bernays in *Grundlagen der Mathematik I* in 1934. Since the non-negative integers are definable in $\langle \mathbb{Z}, +, -, <, 0, 1\rangle$, these decision procedure covered both structures, $\langle \mathbb{Z}, +, -, <, 0, 1\rangle$ and $\langle \mathbb{N}, +, <, 0, 1\rangle$. Skolem and Presburger gave a quantifier elimination algorithm for these theories, but we show here the more efficient algorithm due to Cooper (1970). We actually show the decidability of $Th(\langle \mathbb{Z}, +, -, <, 0, 1\rangle)$, sometimes identified with Presburger's arithmetic (indeed, it has the same expressiveness, since formulas of these two theories are translable each other). Decidability follows from the fact that, given a formula, we can find an equivalent formula quantifier free, and the truth or falsity of such a formula is only a matter of computation. Actually the above theory *does not* satisfy the quantifier elimination, since for instance, the formula $\exists x(y = x + x)$ ("y is even") has no quantifier free equivalent. Indeed, it is provable that for all quantifier free formula $\phi(x)$ in the language of this structure, there is a number n such that for all $m > n$, either $\phi(m)$ or $\neg\phi(m)$ is true in this structure. However, decidability is proved by producing a quantifier elimination algorithm for an *extension* of this theory, obtained by adding a divisibility predicate $k|x$ defined as $\forall x(k|x \leftrightarrow \exists y(x = ky))$. Hence we actually are interested in the structure:

$$\langle \mathbb{Z}, +, -, <, 0, 1, \mathbb{Z}, 2\mathbb{Z}, 3\mathbb{Z}...\rangle$$

The theory of this structure *does* admits the quantifiers elimination and the basic formulas of this language are decidable. However, the theory of the original structure is a subset of the theory of this structure, hence in turn it is itselfs decidable.

We now show the Cooper's method for eliminating quantifiers from this theory, but before we recall this result.

Lemma 27. *Suppose that for every formula of the language of a theory* T *of the form:*

$$\exists x(\alpha_0 \wedge ... \wedge \alpha_n)$$

where each α_i *is atomic or negated atomic, there is a quantifier-free formula provably in* T *equivalent to it: then* T *admits elimination of quantifiers.*

Proof. See Enderton (2001), ch.3.2 and Harrison (2009) 328-49. Thanks to this result we will be concerned only about existential formulas. QED

The algorithm consists of the following steps:

1. Reprocessing $\exists x\phi(x)$, where $\phi(x)$ is quantifier-free:

   (a) by using boolean equivalents and definitions, we reduce the formula to the language $\neg, \wedge, \vee, <, =$.

   (b) replace $\neg(s < t)$ with $t < s + 1$.

   (c) put 0 to the left, so $s = t$ becomes $0 = t - s$ and $s < t$ becomes $0 < t - s$.

   (d) terms are written in canonical form $c_0x_0 + ... + c_nx_n + k$, with the c_i and k integers.

   Hence literals are of the forms

   $$0 = t, \neg(0 = t), 0 < t, k|t, \neg(k|t)$$

   and t is canonical. Atoms containing x are of the form $cx + s$,

2. Compute the least common multiple δ of all coefficients of x in $\phi(x)$ and then replace terms as in the following table, where x has always $\pm\delta$ as coefficients (where $hh' = \delta$):

literal	*is replaced by*
$0 < t - hx$	$0 < h't - \delta x$
$0 < hx - t$	$0 < \delta x - h't$
$k \mid hx + t$	$h'k \mid \delta x + h't$
$\neg(k \mid hx + t)$	$\neg(h'x \mid \delta x + h't)$

(5)

3. Obtain an equivalent formula of the form $\exists x(\psi(x) \wedge \delta \mid x)$ where $\psi(x)$ is $\phi[x/\delta x]$, i.e. we have replaced δx with x. Let us call $\phi^*(x)$ the formula $\psi(x) \wedge \delta \mid x$.

We have now the following alternatives:

1. either $\forall y \exists x < y \phi^*(x)$,

2. or $\exists x \phi^*(x) \wedge \forall y < x \neg\phi^*(x)$

Hence replace $0 = t$ and $0 < t$ with $\perp$, if x occours in t, and replace $0 < t$ with $\top$, if $-x$ occours in t (do nothing on other atoms). Call $\phi^*_\infty(x)$ the result of this replacement.

Lemma 28. *For sufficiently small x the formulas $\phi^*(x)$ and $\phi^*_\infty(x)$ are equivalent. Formally:*

$$\exists y \forall x < y(\phi^*(x) \leftrightarrow \phi^*_\infty(x))$$

Proof. Let us consider the atomic cases where $\phi^*(x)$ is $0 = x + a$ $0 < x + a$. In these cases $\phi^*_\infty(x)$ is $\perp$ and it is true that:

$$\forall x < -a(\phi^*(x) \leftrightarrow \perp)$$

because $x \geq -a$. Hence $y = -a$. In the case $0 < -x + a$, we have that $\phi^*_\infty(x)$ is $\top$ and clearly we have:

$$\forall x < a(\phi^*(x) \leftrightarrow \top)$$

so that $y = a$. As for non atomic cases, let us consider for example the conjunction: if $\phi^*(x)$ is $\beta(x) \wedge \gamma(x)$ and by (IH) we have $\forall x < a(\beta(x) \leftrightarrow \beta_\infty(x))$ and $\forall x < b(\gamma(x) \leftrightarrow \gamma_\infty(x))$. The result follows taking $y = min\{a, b\}$. Other case for exercise. QED

Lemma 29. *$\forall y \exists x < y \phi^*(x)$ is equivalent to $\exists x \phi^*_\infty(x)$.*

Proof. $\Rightarrow$ if $\forall y \exists x < y \phi^*(x)$ holds, then $\phi^*(x)$ holds for arbitrary small values of x, sufficent to make $\phi^*(x)$ and $\phi^*_\infty(x)$ equivalent, according to the previous lemma.

$\Leftarrow$ If $\exists x \phi^*_\infty(x)$ holds, this means that $\phi^*_\infty(n)$ holds for some n. We now show that indeed it holds for infinitely many $x < n$. Note that in a true statement $k \mid \pm x + t$, the formula remains true if x is altered by a multiple of k. Take the LCM δ of all k occurring in formulas of the kind $k \mid s$ in $\phi^*_\infty(x)$, observing that x occurs only in these formulas. Hence subtract to x a multiple fo δ. The truth value of $\phi^*_\infty(x)$ does not change and therefore $\phi^*_\infty(n) \leftrightarrow \phi^*_\infty(n - z\delta)$ for all $z \geq 0$. From this follows $\forall y \exists x < y \phi^*(x)$, by applying the previous lemma. QED

Corollary 16. *$\forall y \exists x < y \phi^*(x)$ is equivalent to $\bigvee_{i=1}^{\delta} \phi^*_\infty(i)$.*

Proof. Since $\forall y \exists x < y \phi^*(x)$ is equivalent to $\exists x \phi^*_\infty(x)$ and $\phi^*_\infty(x)$ is invariant modulo δ (i.e. changing x to $x \pm h\delta$), then $\phi^*_\infty(n)$ holds for some n iff it holds for at least one $i \in [1, \delta]$ (or any other set of δ consecutive integers). This depends on the fact that any n is congruent with some $i \in [1, \delta]$ modulo δ, and therefore $n = k\delta \pm i$.

This as regards the alternative 1. Now let us consider the alternative 2. Let us consider again δ as above (the LCM of all k in formulas $k|t$ of $\phi^*_\infty(x)$). Let i a number such that $\phi^*(i)$ holds, but $\phi^*(i-\delta)$ does not hold. By the invariance of divisibility predicate under δ the change of values of ϕ^* is due to other literals containing x, becoming false when x decreases. QED

We come to the following definition:

Definition 34. *Let $L(x)$ be a literal of ϕ^* containing x , but different from a divisibility predicate. A boundary point for it is given by the following table:*

literal	*boundary point*
$0 = x + t$	$-(t + 1)$
$\neg(0 = x + t)$	$-t$
$0 < x + t$	$-t$
$0 < -x + t$	*none*

(6)

If b is a boundary point for a literal, this literal is false for $x = b$, but true for $x = b + 1$.

Remarks. If there is a minimum i s.t. $\phi^*(i)$, then $\phi^*(i - \delta)$ does not hold.

Theorem 75. *Let δ the LCM as above and let B the boundary set. For all integers i, if $\phi^*(i)$ holds, but $\phi^*(i - \delta)$ does not hold, then there is $b \in B$ such that $1 \leq j \leq \delta$ and $i = b + j$.*

Proof. We check only the base cases, leaving the induction relative to the complex formulas as exercise.

1. If ϕ^* is $(0 = x + t)$, then $i = (-t)$. The boundary point is $-(t + 1)$. Hence there exists $b \in B$ such that $i = b + 1$.
2. ϕ^* is $\neg(0 = x+t)$, the boundary point is $-t$; note that $\phi^*(i-\delta)$ is false only if $i = \delta+b = \delta-t$.
3. If ϕ^* is $0 < x + t$ a boundary point is again $-t$; by assumption $-t + 1 \leq i$ and $i \leq -t + \delta$. Ergo $i = b + j = -t + j$ for some $1 \leq j \leq \delta$.
4. For other kind of literals, it is not possible that $\phi^*(i)$ holds, but $\phi^*(i - \delta)$ does not hold.

Now we compose the alternatives 1. and 2. QED

Corollary 17. *Let $\phi^*(x)$ as above, where all coefficients of x are ± 1. Let δ be the LCM of all k such that $k|t$, for some t containing x occurs in the formula. Then $\exists x \phi^*(x)$ is equivalent to:*

$$\bigvee_{i=1}^{\delta} (\phi^*_\infty(i) \vee \bigvee_{b \in B} \phi^*_\infty(b + i))$$

Proof. By collecting the above results concerning alternatives 1. and 2. $\Rightarrow$ Suppose $\exists \phi^*(x)$ holds. Hence either:

1. $\forall y \exists x < y \phi^*(x)$ or
2. $\exists x \phi^*(x) \wedge \forall y < x \neg \phi^*(y)$

In case (1) we obtain $\bigvee_{i=1}^{\delta} \phi_\infty(i)$ by the previous result. In case (2) there is $i = b + j$ such that $\phi^*(i)$ and $\neg\phi^*(i - \delta)$ and therefore $\bigvee_{j=1}^{\delta} \bigvee_{b \in B} \phi^*(b + j)$.

$\Leftarrow$ If the disjunction holds, then, if $\bigvee_{i=1}^{\delta} \phi_\infty(i)$ then by previous results we have an arbitrarily large negative x with $\phi^*(x)$ and therefore $\exists x \phi^*(x)$. If on the other hand $\bigvee_{j=1}^{\delta} \bigvee_{b \in B} \phi^*(b + j)$ holds, so does $\exists x \phi^*(x)$. QED

Corollary 18. *$Th(\langle \mathbb{Z}, +, -, <, 0, 1, \mathbb{Z}, 2\mathbb{Z}, 3\mathbb{Z}... \rangle)$ is decidable, i.e., there is an effective procedure which, given any sentence of the language of this theory, will decide the truth or falsity of it in $\langle \mathbb{Z}, +, -, <, 0, 1, \mathbb{Z}, 2\mathbb{Z}, 3\mathbb{Z}... \rangle$.*

4.6. Another look at incompleteness: Tennenbaum's theorem

Another kind of "undecidability" theorem is due to Tennenbaum (1959): although the first-order theories of arithmetic are not categorical (see on p.15), it is impossible to build non-standard models of Peano Arithmetic in which the addition and multiplication operations, as well as the order relation are effectively computable. The phenomenon highlighted by Tennenbaum is actually very pervasive and applies to most subtheories of PA. Already Skolem (1955) built nonstandard models of arithmetic by an ultrapower-like construction and had raised the question of recursiveness of nonstandard models. Tennenbaum's theorem improved the result in Mostowski (1957), that no nonstandard model of primitive recursive arithmetic with predicates for all primitive recursive functions can be recursive. On the contrary Shepherdson (1964), using algebraic methods (a model of open induction is an integer part of a real closed field), produced a recursive nonstandard models of arithmetic with axiom schema of induction for quantifier-free formulas (see Kaye (2011) for a more detailed historical background and Berarducci and Otero (1996) and D'Aquino (1997) for further developments).

Definition 35. *Let $\mathcal{U}$ and $\mathcal{B}$ be first-order structures of the same language. A function $f : \mathcal{U} \longrightarrow \mathcal{B}$ is an* homomorphism *iff:*

1. *For all $a_0, ..., a_n \in A$, and functional symbol F, if $F^{\mathcal{U}} = h$ and $F^{\mathcal{B}} = g$, si ha:*

$$f(h(a_0, ..., a_n)) = g(f(a_0), ..., f(a_n))$$

2. *For all $a_0, ..., a_n \in A$, and relational symbol P, if $P^{\mathcal{U}} = R$ and $P^{\mathcal{B}} = \widetilde{R}$, we have:*

$$\langle a_0, ..., a_n \rangle \in R \Rightarrow \langle f(a_0), ..., f(a_n) \rangle \in \widetilde{R}$$

3. *For all constant a_i, if $a_i^{\mathcal{U}} = c_i$ and $a_i^{\mathcal{B}} = b_i$, then $f(c_i) = d_i$.*
   If also the inverse of 2. holds, this is a strong *homomorphism. Also we say that it is an* embedding, *iff*
4. *f is injective: $x \neq y \Rightarrow f(x) \neq f(y)$,*
5. *f is a strong homomorphism.*
6. *It is an elementary embedding iff for all formulas ϕ and all $a_0, ..., a_m \in \mathcal{U}$,*

$$\mathcal{U} \models \phi(a_0, ..., a_m) \; if \; and \; only \; if \; \mathcal{B} \models \phi(j(a_0), ..., j(a_m))$$

We say that f is an isomorphism *(notation $\mathcal{U} \cong_f \mathcal{B}$), iff it is an embedding and it is bijective.*

Related to the algebraic notion of isomorphism, we have the logical notion of *elementary equivalence*: the structures $\mathcal{U}, \mathcal{B}$ are elementarily equivalent (notation $\mathcal{U} \equiv \mathcal{B}$) iff for all *sentence* ϕ, $\mathcal{U} \models \phi \Leftrightarrow \mathcal{B} \models \phi$. Isomorphic structures are also elementarily equivalent, but the reverse implication is not true and we have that:

$$\mathcal{U} \equiv \mathcal{B} \not\Rightarrow \mathcal{U} \cong \mathcal{B}$$

Indeed it is provable that $\langle \mathbb{Q}, < \rangle \equiv \langle \mathbb{R}, < \rangle$, however (Cantor!) $\langle \mathbb{Q}, < \rangle \ncong \langle \mathbb{R}, < \rangle$. A structure which is elementarily equivalent, but not isomorphic, to the standard model is called a *nonstandard model* of arithmetic.

Definition 36. *A model $\mathcal{M}$ for the language of Peano Arithmetic is standard, if it isomorphic to the intended model $\langle \mathbb{N}, +, \cdot, S, 0\rangle$. Otherwise is called non-standard. Let $t^{\mathcal{M}}$ be the element of the model that interprets a term t. The elements x of $\mathcal{M}$ such that $x = \overline{n}^{\mathcal{M}}$ for some number n (i.e. the elements that interpret a natural number) are called standard.*

Theorem 76. *If $\mathcal{M}$ contains a nonstandard element, then it is nonstandard (i.e. non isomorphic to the standard model).*

Proof. Suppose by contradiction that $\mathcal{M}$ is standard but contains a nonstandard element x. Let $g : \mathbb{N} \to \mathcal{M}$ be the alleged isomorphism. Hence for all natural numbers n, $g(\overline{n}^{\mathbb{N}}) = \overline{n}^{\mathcal{M}}$, namely standard elements are mapped in standard elements. But $\mathcal{M}$ contains a non standard element x, and therefore g cannot be suriective, against the hypothesis that it is an isomorphism. QED

We can derive the existence of non-standard models from the compactness. For example, let $\mathcal{L}_{\mathsf{PA}}$ be the language of Peano arithmetic and let us consider the extended language ${\mathcal{L}_{\mathsf{PA}}}^* = \mathcal{L}_{\mathsf{PA}} \cup \{c\}$ and the theory:

$$T^* = \mathsf{PA} \cup \{c > \overline{n} | n \in \mathbb{N}\}$$

where:

$$\overline{n} = \overbrace{S(S(...S(\overline{0})...))}^{n-times}$$

Note that each finite subset of this theory is contained in some:

$$T_k := \mathsf{PA} \cup \{c > \overline{n} | n < k\}$$

But such a T_k is true in every structure $\mathcal{N}_k = \langle \mathbb{N}, k\rangle$, where $c^{\mathcal{N}} = k$. Hence every finite subset of T^* has a model and therefore also T^* has a model $\mathcal{M}$.

However in this model, we have $c^{\mathcal{M}} > k$ for all natural k; it will contain thus at least one non-standard element. In particular this means that the reduct of this model to our original language of PA is not isomorphic to the standard model. Applying Löwenheim-Skolem, we can take the model in question, at most countable. Rather, $\mathbb{N}$ constitutes an initial segment of each non-standard model; non-standard elements come "after", since both the standard model and non-standard, fulfils the following principle, for all $k \in \mathbb{N}$:

$$\forall x(x < \overline{k} \to (x = \overline{0} \vee x = \overline{1} \vee ... \vee x = \overline{k-1}))$$

There are exactly $2^{\aleph_0}$ non isomorphic countable models of arithmetics.

Theorem 77. *There are exactly $2^{\aleph_0}$ non isomorphic countable models of Peano arithmetic* PA.

Proof. Clearly there are *at most* (since there are continuum-many interpretations of $+$, continuum-many interpretations of $\times$ etc.). We can verify that there are *at least*, considering that if ν is Gödel's undecidable sentence relative to PA, then $S_0 = PA + \nu$ and $S_1 = PA + \neg\nu$ they are both consistent and they will therefore have a model; moreover, they will also have respectively a undecidable Gödel's sentence ν_0 and ν_1. Hence we can consider $S_{00} = S_0 + \nu_0$, $S_{01} = S_0 + \neg\nu_0$, $S_{10} = S_1 + \nu_1$, $S_{11} = S_1 + \neg\nu_1$ etc. in general,for all finite binary string τ, $S_{\tau 0} = S_\tau + \nu_\tau$ and $S_{\tau 1} = S_\tau + \neg\nu_\tau$. Recall that $|\{0,1\}|^{\mathbb{N}} = 2^{\aleph_0}$. Each branch of the binary tree corresponds to a completion of PA. Being all these theories consistent and inequivalent, each one has a model, which is not isomorphic to any model of the others. In addition these will be all models of PA. QED

In spite of the apparent chaotic nature of the non-standard part of nonstandard models, it actually has a very precise order type.

Definition 37. *If $\langle P, <_P\rangle$ and $\langle Q, <_Q\rangle$ are linear orders, let:*

1. *$P + Q = P \times \{0\} \cup Q \times \{1\}$ the sum of them, where $\langle a, i\rangle <_{P+Q} \langle b, j\rangle$ if and only if $j = i = 0$ and $a <_P b$, or $j = i = 1$ and $a <_Q b$, or $i = 0$ e $j = 1$ (i.e. the elements of P and* Q *maintain their original order, but the elements in P precede those in* Q*).*

2. *let $P \times Q$ be the product of them, and order it lexicographically $\langle a, b\rangle <_{P\times Q} \langle c, d\rangle$ iff either $a <_P c$ or $a = c$ and $b <_Q d$.*

Theorem 78. (Henkin 1950) *Each non-standard countable model $\mathcal{U}$ of Peano arithmetic has order type $\mathbb{N} + \mathbb{Q} \times \mathbb{Z}$.*

Proof. Now we observe that $\mathcal{U}$ has $\mathbb{N}$ as initial segment and then its order will be of the form $\mathbb{N} + A$, for some linear order A.

Let $a, b \in \mathcal{U}\backslash\mathbb{N}$; we define the following equivalence $a \sim b \Leftrightarrow |a - b| \in \mathbb{N}$, namely, a, b are in this relation, iff the absolute value of the difference is standard. Moreover, let $a \prec b$ iff $a < b$ and $|a - b| \notin \mathbb{N}$ ("b is much bigger than a"). Lastly:

$$[a] \prec^* [b] \Leftrightarrow a \prec b$$

Observe that a is non-standard, then $[a]_\sim = \{a - n, a + n | n \in \mathbb{N}\}$, namely, the equivalence class of a has the form of $...a - 2, a - 1, a, a + 1, a + 2...$ Note that this order is isomorphic to that of integers $\mathbb{Z}$. Also notes that the following holds:

1. $\prec$ is transitive and antireflexive.
2. $a \prec b, c \sim a, d \sim b \Rightarrow c \prec d$
3. $a \prec b \Rightarrow \neg(b \prec a)$

Lastly, note that $\prec^*$ lacks a maximum, lacks a minimum, and is also a dense order, since the following hold:

1. $[a] \prec^* [a + a]$ (no last element)
2. $[a/2] \prec^* [a]$ *or* $[a + 1/2] \prec^* [a]$ (no first element)
3. $[a] \prec^* [b] \Rightarrow [a] \prec^* [a + b/2] \prec^* [b]$ (density)

But this order is isomorphic to that of rational $\mathbb{Q}$ (remember that for a result of Cantor, *all dense countable linear orders without endpoints are isomorphic*); hence, consider that $...\xi_{q_0}, \xi_{q_1}, \xi_{q_2}...$ are the equivalence classes with respect to $\sim$, ordered by $\prec^*$, that is isomorphic to the order of $\mathbb{Q}$, while inside them, they are ordered as $\mathbb{Z}$. Hence the equivalence classes $\xi_{q_0} \prec^* \xi_{q_1} \prec^* \xi_{q_2}...$ are ordered like $\mathbb{Q}$ and each one has the order of $\mathbb{Z}$:

$$\xi_q = ...a_{q_{z-2}} < a_{q_{z-1}} < a_{q_z} < a_{q_{z+1}} < a_{q_{z+2}}...$$

then the elements a_{q_z} of the non-standard part of $\mathcal{U}$ can be ordered in this way:

$$a_{q_z} <_U a_{q'_{z'}} \Leftrightarrow q <_\mathbb{Q} q' \vee (q =_\mathbb{Q} q' \wedge z <_\mathbb{Z} z')$$

This, according to the above definition of product between orders, is the order type of $\mathbb{Q} \times \mathbb{Z}$. So the whole model has the order type $\mathbb{N} + \mathbb{Q} \times \mathbb{Z}$. QED

For a remark of Potthoff (1969), in the previous theorem, we will see now, we cannot replace $\mathbb{Q}$ with $\mathbb{R}$, for any non-standard model. To see this, a key result of this field is needed. First observe that the standard part of a non standard model M, is not definable in M. Suppose by contradiction that it is, i.e. that for some $\sigma(x)$, $\mathcal{M} \models \sigma[a]$ (i.e. a satisfies σ in the model $\mathcal{M}$) iff $a \in \mathbb{N}$, and therefore $\neg\sigma(x)$ will define the non standard part $\mathcal{M} \setminus \mathbb{N}$: but then there will be a minimum (the principle of the minimum number is equivalent to induction) non-standard, that is contradiction, as we saw about the structure of non-standard models.

Hence the following obtain.

Theorem 79. (Overspill) *If $\phi[n]$ holds for all $n \in \mathbb{N}$, then exists $a \in M \setminus \mathbb{N}$, such that $\phi[b]$ holds for all $b \leq a$.*

Proof. Indeed, if no a non standard satisfied $\phi[a]$, then the formula $\sigma(x)$ defined as $\exists y(x < y \wedge \phi(y))$ would define the standard part, against what we said. In other words, if $\mathcal{M} \models \sigma[a] \Leftrightarrow a \in \mathbb{N}$, then in particular, since $0 \in \mathbb{N}$ and if $n \in \mathbb{N}$, also $n+1 \in \mathbb{N}$, we would have $\mathcal{M} \models \sigma(\overline{0}) \wedge \forall x(\sigma(x) \to \sigma(x+1))$. However, being $\mathcal{M}$ a model of PA it must satisfy the induction principle, from which follows $\mathcal{M} \models \sigma(\overline{a})$, for all a, included non standard elements. QED

An interesting application (see Bovykin and Kaye (2002)) of the *overspill* theorem is the proof of the above-mentioned fact that in the order type $\mathbb{N} + \mathbb{Q} \times \mathbb{Z}$, is not possible to replace $\mathbb{Q}$ with $\mathbb{R}$; in other words we cannot consider the equivalence classes $[a] = \{...a-2, a-1, a, a+1, a+2...\}$ as "reals". Recall that the real numbers fulfill the so-called *Dedekind completeness property*, that can be also expressed in this form:

> Let X be a set of real numbers; an upper bound for X is a real r such that $r \geq a$, for all $a \in X$. The least upper bound property says that any non-empty set of reals having an upper bound, has a least upper bound.

Now we just apply this principle.

Let us take a non standard model $\mathcal{M}$, an element a nonstandard of the model and a sequence $\{ia\}_{i \in \mathcal{M}}$. Note that $i < j$, where i, j are standard, implies

$$[ia] \prec^* [ja]$$

Suppose there is a least upper bound $[b] = sup\{[na] | n \in \mathbb{N}\}$. Then, for all $n \in \mathbb{N}$, $na \prec b$ and therefore by overspill, there will be an element c non standard such that $ca \prec b$. Hence, for all $n \in \mathbb{N}$, $na \prec ca \prec b$. But all reals $\{[sa]\}_{\mathbb{N}<s<c}$ are between $\{[na] | n \in \mathbb{N}\}$ and $[b]$ (contradiction, against the hypothesis that $[b] = sup\{[na] | n \in \mathbb{N}\}$).

We formulate this result with respect to PA, but, as said, it appropriately extends to all its subtheories, until minimum extensions of the *open induction* theory IOpen, for which, as proved for the first time by Sheperdson, there exist recursive models. It states that in these theories, the standard model is the only one in which the operations of addition and multiplication are recursive. This can be seen as a kind of "incompleteness result", which states that we can not build a non standard model of Peano arithmetic.

Definition 38. *The "standard system" $SSy(\mathcal{M})$ of a non-standard model $\mathcal{M}$ is the set of subsets $A \subseteq \mathbb{N}$ such that $n \in A$ iff $\mathcal{M} \models \phi(a, n)$, for some $\phi(x, y)$ and some $a \in \mathcal{M}$.*

Remember that in PA is Σ^0_1-definable the primitive recursive predicate "p is the x^{th}-prime". Note that $\mathbb{N}$ and $\mathcal{M}$ agree on the standard prime numbers, in the sense that p_n, for $n \in \mathbb{N}$ is the n^{th}-prime in both models. It is not hard to show that the "standard system" can actually be equivalently defined also as the set of subsets $S \subseteq \mathbb{N}$ such that: $S = \{n \in \mathbb{N} | \mathcal{M} \models p_n | a\}$ for some a of the model.

Theorem 80. (Tennenbaum 1959) *Let $\mathcal{M} = \langle M, \oplus, \otimes, S, 0\rangle$ a non standard countable model of* PA *non isomorphic to the standard model. Then $\mathcal{M}$ is not recursive.*

Proof. We show first that $SSy(\mathcal{M})$ contains a non recursive set. Let indeed $A, B \subseteq \mathbb{N}$ disjoint recursively enumerable and recursively inseparable. Being computably enumerable they will be definable in $\mathbb{N}$ respectively by Σ^0_1 formulas $\exists y \alpha(x, y)$ and $\exists y \beta(x, y)$; but these are preserved in the extensions of $\mathbb{N}$. In particular, since A and B have empty intersection, we will have for each k:

$$\mathbb{N} \models \forall x < \overline{k} \forall y < \overline{k} \forall z < \overline{k} \neg(\alpha(x, y) \wedge \beta(x, z))$$

But by "overspill" this holds also in M with some $a \in M \setminus \mathbb{N}$.

If now we define $C = \{n \in \mathbb{N} | \mathcal{M} \models \exists y < a \alpha(n, y)\}$, then we note that $A \subseteq C$ and $C \cap B = \emptyset$. Hence, being $A, B \subseteq \mathbb{N}$ recursively inseparable, $C \in SSy(\mathcal{M})$ but it cannot be recursive. QED

Given $C \in SSy(\mathcal{M})$, that we assume not recursive, it can be coded in $\mathcal{M}$ in this way: $n \in C$ if and only if $\mathcal{M} \models \exists y(c = y \otimes p_n)$, for some $c \in M$, where p_n is the n-th prime number. We show that $\oplus$ is not recursive. It is crucial that, being b and p_n standard, we can express $(y \otimes p_n) \oplus b$ as $y \oplus y \oplus ... \oplus y \oplus 1 \oplus ... \oplus 1$. Let us consider therefore the disjunction of the following formulas:

$$(c = \overbrace{y \oplus ... \oplus y}^{p_n - times}), (c = \overbrace{y \oplus ... \oplus y}^{p_n - times} \oplus 1), ..., (c = \overbrace{y \oplus ... \oplus y}^{p_n - times} \oplus \overbrace{1 \oplus ... \oplus 1}^{p_n - 1 - times})$$

Since it is a model of PA, the non standard model $\mathcal{M}$ will verify the "Euclidean division":

$$\forall x \forall z(z \neq 0 \rightarrow \exists y \exists b(x = yz + b \wedge 0 \leq b < y))$$

Fix $z = p_n$ and $x = c$. Hence in the model there are (unique) y and $0 \leq b < p_n$ such that $c = (y \otimes p_n) \oplus b$.

Since p_n is standard, the theory proves:

$$\forall x(x < \overline{p_n} \leftrightarrow \bigvee_{k < p_n} x = \overline{k})$$

so that we have these alternatives:

1. If $b = 0$, then $c = (a \otimes p_n^M)$, hence the disjunction is true, since the first disjunct is true, and therefore $\mathcal{M} \models \exists y(c = y \otimes p_n)$, from which $n \in C$.

2. If $b = 1 \oplus 1 \oplus ... \oplus 1$, for some number of summands less than p_n. Then $n \notin C$, because is true one of the other disjucts and therefore $\mathcal{M} \not\models \exists y(c = y \otimes p_n)$.

But then, if $\oplus$ is recursive, in $\mathcal{M}$ is decidable if either $n \in C$ or $n \notin C$. Given n, compute p_n and search for the unique $y \in \mathcal{M}$ such that:

$$(\overbrace{y \oplus ... \oplus y}^{p_n - times}) \oplus \overbrace{1 \oplus 1 \oplus ... \oplus 1}^{b - times} = c$$

Indeed, our search is guaranteed to terminate, being b and y uniquely determined by n and c . Hence, at the end, if $b = 0$, then we know that $n \in C$, otherwise, we know that $n \notin C$. Contradiction, because C was not recursive. The case of $\otimes$ is treated in a similar way.

A more refined version of Tennenbaum's theorem can be expressed in this form (see Smorynski (1984)).

Theorem 81. *In a nonstandard model $\mathcal{M} = \langle \mathbb{N}, \otimes, \oplus, S, 0 \rangle$, every set X in the standard system is recursive in each $\otimes$ and $\oplus$.*

Proof. Recall that $n \in X$ iff $\mathcal{M} \models \phi(n, b)$, for some $b \in \mathcal{M}$ and formula ϕ, iff for some $b \in \mathcal{M}$, $\mathcal{M} \models p_n | b$, iff $\exists c(p_n^{\mathcal{M}} \otimes c = b)$ iff $\exists c(c \oplus ... \oplus c = b)$ (p_n-times). If we call the (by hypothesis, recursive) relation expressed in parentheses "A", then this means that X is Σ_1^A definable, in the relativized arithmetical hierarchy, and therefore X is r.e. in A (by the relativised version of a well-known result). Since the complement of X correspond to $\neg\phi$ the same argument leads to the conclusion that also the *complement* of X is r.e. in A and therefore X is recursive in A. Analogously, for multiplication, if c, b are as above, take $d = 2^c$ and $e = 2^b$; hence $d \otimes ... \otimes d = e$, because $2^c \otimes ... \otimes 2^c = 2^{c \oplus ... \oplus c}$ and the same argument apply, so that X is also recursive in $\otimes$. Tennenbaum's theorem follows: if these operations $\otimes$ and $\oplus$ were computable, also all elements of $SS(\mathcal{M})$ would be computable, because if A is computable and $X \leq_T A$, then X is computable too; but we have shown that this is false, because there exist non computable sets in the standard system.
QED

4.7. Tennenbaum's and Gödel's theorems

We would now investigate the relationship between Tennenbaum's and Gödel's theorems, showing how to use Tennenbaum's result to obtain the incompleteness theorem. A proof of the completeness theorem for countable vocabularies can be based on König's lemma ("every finitely branching infinite tree has an infinite branch"). Refinements of König's lemma are possible, e.g. the so-called "Kreisel basis theorem", according to which every infinite computable binary tree has an infinite path *recursive in* $\emptyset'$, i.e.. computable from the halting problem, or, equivalently, by Post's theorem, a Δ_2 path.

These refinements yield refinements to the first-order Completeness Theorem, leading to the conclusion that that there exists a model $\mathcal{M}$ of PA where $\otimes, \oplus$ are Δ_2-definable. More precisely, let S_M be the set of pairs $\langle \ulcorner \phi \urcorner, n \rangle \in \mathbb{N} \times \mathbb{N}$ such that $\mathcal{M} \models \phi(\overline{n})$. Say that $\mathcal{M}$ *is a* Δ_2 *model* if S_M is Δ_2 definable in the standard model. In other words, a Δ_2 model, is a model $\mathcal{M}$ with domain $\mathbb{N}$ such that the set of formulas (with parameters) which are true in the model is Δ_2-definable. Then the *Arithmetized Completeness Theorem* says that each consistent recursively axiomatizable theory has a Δ_2 model $\mathcal{M}$ (see Berarducci and Mamino (2023), Kaye (1991), Kennedy (2022) pp. 153-188, Kossak and Schmerl (2006), p. 163 for further details on all the points mentioned).

Hence, by the generalized Tennenbaum theorem above and Post theorem, every set in $SS(\mathcal{M})$ is Δ_2-definable too: indeed, by Post's theorem, $A \in \Delta_2$ iff $A \leq_T \emptyset'$ and therefore, by transitivity of the reduction, if $B \leq_T A$, then also $B \in \Delta_2$. Now suppose by contradiction that the model of the arithmetized completeness theorem $\mathcal{M}$ is an *elementary extension* of the standard model (in symbols $\mathbb{N} \preceq \mathcal{M}$): this means that for every formula $\phi(x)$ and every element $b \in \mathbb{N}$, this b satisfies the formula in $\mathbb{N}$ iff it satisfies the formula in $\mathcal{M}$. Then *all* arithmetically definable sets, i.e. all Σ_n sets for every n are in $SS(\mathcal{M})$ and at the same time are computable from the $\otimes, \oplus$ of this model. Therefore these operations cannot be Δ_2-definable or arithmetical at all (contradiction). Hence $\mathbb{N} \not\preceq \mathcal{M}$ and therefore there is some sentence ψ true in $\mathbb{N}$ but false in $\mathcal{M}$ from which follows PA $\nvdash \psi$.