6. Independent sentences of mathematical character

6.1. Skepticism about Gödel's results

It has been pointed out (see for instance Grattan-Guinness (2011)) that the reception of incompleteness results within the *mathematical* community was very slow. One reason for the large underestimation of gödelians' results in part of the community of mathematicians, which helped to brake their assimilation, was linked to the *metamathematical* character of the statement "I am not provable" used in the constructive proof. The perceived distance from the concrete mathematical work is perhaps behind the most striking case of the silence in this regard: that of the the French group 'Bourbaki' of formalists mathematicians that began its activity in 1935. Actually still in 1948 Dieudonné, in David Hilbert's obituary, could write rather vaguely: "it seems that Hilbert's intuition ... has resulted in hopes a little exaggerated". Moreover, judging the presence of an undecidable proposition to be irrelevant, in Dieudonné (1987) the French mathematician complains that the undecidable statement of Gödel's first theorem is too artificial and unrelated to number theory. Nor were the mathematicians of the Bourbaki group too concerned about the problem of consistency, that considered just an empirical fact.

On the one hand, the formalization of mathematical reasoning, and on the other hand, the self-referential arithmetical statement, seemed to many mathematicians to be builded ad hoc for the purpose of obtaining the incompleteness result and very different from those occurring naturally in the mathematical research. In other words, the working mathematicians of the 1930s, although impressed by the wide-ranging philosophical implications of Gödel's limitative result, need not themselves have felt particularly limited by them, and could continue in their research for the most part as before.

A question that naturally arised, was therefore that about the pervasivity of Gödel's results, its impact on mathematical practice and if there were statements coming from the concrete mathematical practice, who shared the same fate of the bizarre statement invented by Gödel, but without resorting to diagonalization and other tricks. To a certain extent the underestimation of Gödelian achievements conceals a misunderstanding. Kurt Gödel himself, as some sources report, reacted to the objection of the *logical*, rather than *mathematical*, character of his results by saying that nothing is more mathematical than a Diophantine equation (see Kripke (2021) and the discussion on Matiyasevich's theorem on p.46). The tools introduced by the incompleteness theorems are perhaps better defined as "strikingly original mathematics, with something of the charm of Cantor's first work in set theory", as remarks Macintyre (2011), although, according to this prominent British model-theorist, the discovery of the phenomenon of incompleteness had little effect on current mathematics.

For this reason, in an attempt to address these objections, some scholars have devoted themselves to the search for independent statements of mathematical content, with proofs that did not rely on the typical Gödelian toolbox. These statements were actually found systematically, starting from Paris (1978), both of combinatorial as Paris and Harrington (1977), or numeric character, as for example Kirby and Paris (1982), i.e. expressing natural

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properties of integers. The distinguished logician John Barwise, editor of the famous Barwise and Keisler (1977), points out at p. 1133 that as early as 1931, i.e. the year of the publication of the incompleteness theorem, the mathematical world was already calling for similar results concerning statements of a clear and simple mathematical nature. Barwise points to the Paris and Harrington theorem as the first example of this kind (in the authors' own words: "a reasonably natural theorem of finitary combinatorics, a simple extension of the Finite Ramsey Theorem"). The other popular example is Paris and Kirby's independence result concerning Goodstein sequences and of the related 'Hydra game'. We give an account of both in this chapter. Goodstein (1944) proved the rather surprising result that eventually these sequences reach 0. Studying the correspondence between Bernays and Goodstein, Rathjen (2015) shows how close Goodstein came to proving the independence result later obtained in Kirby and Paris (1982). We will see that this last paper actually showed that the power of Peano's Arithmetic is not enough to prove this result also for a kind of sequences so-called *special* Goodstein sequences. Indeed, it has been emphasised by various logicians that the first strictly mathematical result of independence was actually the famous result of independence of the induction principle up to ε_0 of Gentzen (1936). In particular Kripke (2021) puts the question in these terms: Gentzen gave the first such result, that was restated by Goodstein in a number-theoretic form.

6.2. Ramsey's theorems and the Paris-Harringhton theorem

An important example of combinatorial statement formalized in first order Peano arithmetic, but independent of it, arises in the context of the mathematical study of combinatorial objects known as *Ramsey theory*. In a concise manner, we are concerned with problems like that posed in *The American Mathematical Monthly* in 1958 (N. 65, vol. 6, Problem E 1321):

> "Prove that in a meeting of any six persons, about three of them are either mutual acquaintances or complete strangers to each other".

The party acquaintances problem. So, what is the minimum number of guests such that it is sure to find among them three people who do not know each other, or three people who know each other? For example, let us suppose that there are five individuals: observe the complete graph K5 below, where the red lines connect individuals who do not know each others and the blue lines individuals who they know to persuade you that there cannot be three people who know each other or three people who do not know:

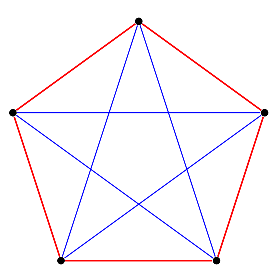

Figure 5. Graph K5.

But if we add an individual (the complete graph K6 below), then six individuals are sufficient to determine a *clique* of three individuals who either know or do not know each other:

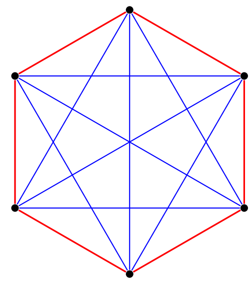

Figure 6. Graph K6.

Notice that in this case each node can belong to at least three red edges, or at least three blue edges. For each node x there are 5 edges incident to it. Take for example three of these edges (x, a), (x, b) and (x, c) of the same color, say blue. Now, if any of the edges $(a, b), (a, c)$ and (b, c) is blue, then we're done. If not, then all these last three edges must be red and we are done as well.

The general problem is: find the minimum number $R(k, h)$ of guests which is necessary to invite to a party in such a way that at least k of them know each other, or at least h guests do not know each other. In our example, $R(3, 3) = 6$.

Ramsey's number $R(h, k)$ in general is definde as the minimum n such that for any coloring in two colors B and R of the edges of the complete graph (i.e. in which each pair of nodes is connected with an arc) with n-nodes, the graph contains, either an R-subgraph with k-nodes, or a B-subgraph with h-nodes, where a coloration is a function $f : \{\langle x, y\rangle | x, y \leq n\} \rightarrow \{B, R\}$.

The *Ramsey theorem for graphs* states that for all k, h there is a n such that each complete graph of at least n nodes and colored with two colors *Blue*, *Red*, must contain a *Blue* monocromatic subgraph with k nodes, or a *Red* monocromatic subgraph with h nodes. This result holds also for a finite number of colors. The *infinite* version states that every countably infinite complete graph must contain a monochromatic complete infinite graph. In view of the results we wish to present, we observe that the theorem can be given in a more general version. First let $[A]^k = \{X \subseteq A || X| = k\}$, where with the notation "$|X|$" we mean the cardinality of X; notice that a graph $G = \langle V, E\rangle$ is given by a set of nodes V and a set of edges E, namely of unordered pairs of nodes, that can be seen as a subset of $[V]^2$ (a complete graph is one such that $E = [V]^2$). E.g. $A = \{a, b, c, d, e\}$, then a three-color (say red, blue and green) coloring of $[A]^4$ might be the following:

(r) $\{a, b, c, d\}$, $\{a, b, d, e\}$

(b) $\{a, c, d, e\}$

(g) $\{a, b, c, e\}$, $\{b, c, d, e\}$.

Ramsey's theorem for sets. Given $X \subseteq \mathbb{N}$, we write $[X]^k$ to indicate the $Y \subseteq X$ such that $|Y| = k$. Moreover, by identifiyng a number with the set of its predecessors $m = \{0, 1, 2, ..., m-1\}$, a surjective function $f : [X]^k \longrightarrow m$ will be called *coloring*, or *partition*. A subset $Y \subseteq X$ will be called *omogeneous*, or *monocromatic*, if all elements of $[Y]^k$ receive the same color by f, i.e. if $f\restriction_Y$ is costant. Lastly, with:

$$n \longrightarrow (h)^k_m$$

we mean that for all set X of cardinality $\geq n$, for all coloring $f : [X]^k \longrightarrow m$, there is a $Y \subseteq X$ of cardinality $\geq h$ such that Y is homogeneous with respect to f.

Theorem 98. (Infinite Ramsey theorem) *Let n, k be positive integers and let $X \subseteq \mathbb{N}$ be infinite. If $[X]^n$ is colored with k colors, then X has a monocromatic subset. In symbols:*

$$\aleph_0 \longrightarrow (\aleph_0)^n_k$$

Proof. We follow Marker (2002) Chapter 5.1. Induction on n:

(a) $n = 1$. Notice that $[X]^1 = \{\{x\}|x \in X\}$. Hence, if $f : [X]^1 \longrightarrow K$, then there is an infinite $Y \subseteq X$ such that f is constant on $[Y]^1$, i.e. there exists $i < k$ such that $f^{-1}(i)$ is infinite. This follows from the "Pigeon hole principle": if we place an infinite number of objects in a finite number of boxes, at least one box will contain infinite objects.

(b) $n > 1$. Without loss of generality, set $X = \mathbb{N}$. Let $f : [\mathbb{N}]^n \longrightarrow k$. Suppose the theorem holds for $n - 1$ by inductive hypothesis. Let us define a sequence:

$$\mathbb{N} = X_0 \supset X_1 \supset X_2 \supset ...$$

and a sequence $a_0 < a_1 < a_2 < ...$ where $a_i = min(X_i)$. Suppose we have defined a_s and X_s and let:

$$f_{a_s} : [X_s \setminus \{a_s\}]^{n-1} \longrightarrow k$$

defined as:

$$f_{a_s}(A) = f(A \cup \{a_s\})$$

Hence $A \cup \{a_s\} = \{a_s, a_{i_0}, ..., a_{i_{n-2}}\} \subseteq X_s$. By the induction hypothesis exists $Z \subseteq X_s \setminus \{a_s\}$ homogeneous with respect to f_{a_s}. Hence we let $X_{s+1} = Z$.

Let now C_{a_s} be the color assigned by f_{a_s} to all elements of $[Z]^{n-1}$. Notice tha colors are *finite*, while the a_s are *infinite*, hence for an infinite $Y = \{a_{m_0}, a_{m_1}, a_{m_2} ...\}$ we will have $C_{a_{m_0}} = C_{a_{m_1}} = C_{a_{m_2}} = ... = j$. Then, for all $\{a_{m_{j_0}}, ..., a_{m_{j_{n-1}}}\} \in [Y]^n$, $f(\{a_{m_{j_0}}, ..., a_{m_{j_{n-1}}}\}) = j$. Indeed, by definition $f(\{a_{m_{j_0}}, ..., a_{m_{j_{n-1}}}\}) = f_{a_{m_{j_0}}}(\{a_{m_{j_1}}, ..., a_{m_{j_{n-1}}}\} = j$, where $\{a_{m_{j_1}}, ..., a_{m_{j_{n-1}}}\} \subseteq X_{m_{j_0+1}}$. QED

Corollary 20. (Finite Ramsey theorem) *For all $k, n, m \in \mathbb{N}$ there exists $h \in \mathbb{N}$ such that:*

$$h \longrightarrow (m)^n_k$$

Proof. Suppose this is not true. Hence for all $h \in \mathbb{N}$, let T_h the set of colorations $f : [h]^n \longrightarrow k$ such that there is no $X \subseteq h$, of cardinality bigger or equal to m, X monocromatic for f. Notice that if $f \in T_{h+1}$, then there is a unique $g \in T_h$ such that $g \subset f$. Let therefore $T = \bigcup_h T_h$;it is a finitely branching tree but infinite and therefore by König's lemma it has an infinite branch $f_0 \subset f_1 \subset f_2 \subset ...$, where $f_i \in T_i$. Let therefore $f = \bigcup_i f_i$; hence $f : [\mathbb{N}]^n \longrightarrow k$. For the infinite Ramsey theorem exists an infinite set $X <$ homogeneous for f. Let therefore $x_0, .., x_{m-1}$ the first m elements of X and let $s > x_{m-1}$. Then $\{x_0, .., x_{m-1}\}$ will be homogeneous for f_s (contradiction). QED

We would now like to show that changing the premises of the the finite Ramsey theorem in an apparently harmless manner, as the fact that the proof is almost the same invites one to think (but that is not the only one for it!), has unexpected consequences. Notice that the proof that uses the *infinite version* is similar to that of the finite Ramsey theorem.

Theorem 99. *For all $k, n, m \in \mathbb{N}$ exists $h \in \mathbb{N}$ such that, if $f : [h]^n \longrightarrow k$ is a coloring, then there exists $Y \subseteq h$ homogeneous for f and such that:*

(a) $|Y| \geq m$

(b) $|Y| \geq min(Y)$ *("Y is relatively big").*

In symbols $h \longrightarrow_* (m)^n_k$.

Proof. Suppose that not; then for each $h \in \mathbb{N}$, let T_h the set of colorings $f : [h]^n \longrightarrow k$ such that there is no $Y \subseteq h$, of cardinality bigger or equal to m and to $min(Y)$, Y monocromatic for f. Hence let us consider $T = \bigcup_h T_h$; it is an infinite, finitely branching tree and therefore the König's lemma has an infinite branch $f_0 \subset f_1 \subset f_2 \subset ...$ where $f_i \in T_i$. Let therefore $f = \bigcup_i f_i$; hence $f : [\mathbb{N}]^n \longrightarrow k$. For the infinite theorem of Ramsey exists an infinite set X homogeneous for f. Let $x_0 = min(X)$; let $x_0, ..., x_{e-1}$ the first e elements of X and let $s \geq e \geq x_0, m$. Then $Y = \{x_0, ..., x_{e-1}\}$ is homogeneous for f_s, $|Y| \geq m, min(Y)$ (contradiction). QED

Nevertheless, there are considerable differences between these statements:

(a) The infinite Ramsey's theorem is formalized and provable in *Peano Arithmetic* of the second order PA_2.

(b) The finite Ramsey's theorem is provable in first order *Peano Arithmetic* PA.

(c) The Paris-Harrington's theorem is formalized in PA and true in the standard model, but unprovable in PA.

Following Marker (2002), pp. 175-202, we prove point 3. in a manner that is quite usual today, namely by proving the independence of the Kanamori-McAloon principle. This theorem is actually a consequence of Paris and Harrington's statement.

Definition 46. *Let us say that:*

(a) if $f : [X]^n \longrightarrow \mathbb{N}$ is a coloring, it is regressive when $f(A) < min(A)$ for all $A \in [X]^n$

(b) $Y \subseteq X$ is called min-homogeneous for f, if when $A, B \in [Y]^n$ and $min(A) = min(B)$, then $f(A) = f(B)$.

For instance $f : [\{1, 2, 3\}]^2 \longrightarrow 2$ where $f(\{1, 2\}) = f(\{1, 3\}) = 0$ and $f(\{2, 3\}) = 1$ is regressive and the set $\{1, 2, 3\}$ is min-homogeneous for f, because $min(\{1, 2\}) = min(\{1, 3\})$ and $f(\{1, 2\}) = f(\{1, 3\})$.

Theorem 100. *For all $c, m, n, k \in \mathbb{N}$ exists d such that, if $f_1, ..., f_k : [d]^n \longrightarrow d$ are regressive, then exists a subset $Y \subseteq [c, d)$ such that:*

(a) $|Y| \geq m$

(b) Y is min-homogeneous for all $f_1, ..., f_k$.

Although formalizable in the language of PA *and true, this statement is not provable in* PA.

Definition 47. *Let:*

$$\Gamma = \{\phi_1(u_1, ..., u_m, v_1, ..., v_n), ..., \phi_e(u_1, ..., u_m, v_1, ..., v_n)\}$$

a set of formulas and let $\mathcal{M}$ be a model of PA. *Let us call $I \subseteq M$ a set of indiscernible elements for Γ, if for all set of elements $x_0 < x_1 < ... < x_n$ e $x_0 < y_1 < ... < y_n$ in I, for all $a_1, ..., a_m < x_0$ and all $\phi_i \in \Gamma$, we have*[1]

From Kanamori-McAloon follows the existence of a set of indiscernibles in the *standard* model.

Main Lemma. For all $e, m, n \in \mathbb{N}$ and formulas;

$$\phi_1(u_1, ..., u_k, v_1, ..., v_n), ..., \phi_e(u_1, ..., u_k, v_1, ..., v_n)$$

there exists a set of indiscernibles I, such that $|I| \geq m$.

1 To improve readability, we use this simplified notation $\mathcal{M} \models \psi(a)$, where a is an element of the model (not a constant or a numeral), meaning that a satisfies $\phi(x)$ in that model.

Proof. Let $m > 2n$. We will use two facts:

(a) From the finite version of Ramsey theorem, let us take w such that:

$$w \longrightarrow (m+n)^{2n+1}_{e+1}$$

(b) From Kanamori and McAloon, given $w, 2n+1$, there exists s such that for all regressive functions $f_1, ..., f_k : [s]^{2n+1} \longrightarrow s$ there exist a subset $Y \subseteq s$ such that $|Y| \geq w$ and Y is min-homogeneous for all $f_1, ..., f_k$.

Hence let us define *regressive functions* $f_1, ..., f_k$ and a coloring g in this way. Let $X = \{x_0, ..., x_{2n}\}$ where $x_0 < ... < x_{2n} < s$. Then:

(a) if for all $0 < i \leq e$ and $a_1, ..., a_k < x_0$ we have:

$$\phi_i(a_1, ..., a_k, x_1, ..., x_n) \leftrightarrow \phi_i(a_1, ..., a_k, x_{n+1}, ..., x_{2n})$$

then let $f_j(X) = 0$, for all $0 < j \leq k$ and $g(X) = 0$.

(b) If there are instead $i \leq e$ and $a_1, ..., a_k < x_0$ such that:

$$\neg(\phi_i(a_1, ..., a_k, x_1, ..., x_n) \leftrightarrow \phi_i(a_1, ..., a_k, x_{n+1}, ..., x_{2n}))$$

then let $f_j(X) = a_j$, for all $0 < j \leq k$, and $g(X) = i$. In this way:

$$\neg(\phi_{g(X)}(f_1(X), ..., f_k(X), x_1, ..., x_n) \leftrightarrow \phi_{g(X)}(f_1(X), ..., f_k(X), x_{n+1}, ..., x_{2n}))$$

Note that $f_j(X) < min(X)$, indeed, either $f_j(X) = 0$, or $f_j(X) = a_j < x_0 = min(X)$, namely the $f_j : [s]^{2n+1} \to s$ are *regressive*. Therefore we can apply Kanamori - McAloon to find $Y \subseteq s$ min-homogeneous, $|Y| \geq w$; for Ramsey and the choice of w there exist $Z \subseteq Y$ and $i < e+1$ such that $g(A) = i$, for all $A \in [Z]^{2n+1}$, $|Z| > m+n$ (recall that $w \longrightarrow (m+n)^{2n+1}_{e+1}$ and that $g : [s]^{2n+1} \longrightarrow e+1$).

Recall that $m > 2n$ and then $|Z| > 3n$, so that we can write Z as $x_0 < x_1 < ... < x_{3n} < ...$. Note that Z is min-homogeneous for each f_j.

Claim. We show now that $g(A) = i = 0$ for all $A \in [Z]^{2n+1}$. Let therefore:

(a) $\alpha = x_0, x_1, ..., x_n, x_{n+1}, ..., x_{2n}, \overline{x_{2n+1}, ..., x_{3n}}$

(b) $\beta = x_0, x_1, ..., x_n, \overline{x_{n+1}, ..., x_{2n}}, x_{2n+1}, ..., x_{3n}$

(c) $\gamma = x_0, \overline{x_1, ..., x_n}, x_{n+1}, ..., x_{2n}, x_{2n+1}, ..., x_{3n}$

subsets of Z of cardinality $2n+1$, where we highligth the omitted elements.

Suppose by contradiction that $i > 0$. Note that α, β, γ are subsets of Z whose cardinality is $2n+1$. Let now $a_j = f_j(\alpha) = f_j(\beta) = f_j(\gamma)$ observing that from the min-homogeneity the sequences α, β, γ of elements of Z give the same result for each f_j and, as observed before, this is less than x_0. For Ramsey's theorem (being Z homogeneous for g) $g(\alpha) = g(\beta) = g(\gamma) = i > 0$. Condition (2) says that:

(a) $\neg\phi_i(a_1, ..., a_k, x_1, ..., x_n) \leftrightarrow \phi_i(a_1, ..., a_k, x_{n+1}, ..., x_{2n})$

(b) $\neg\phi_i(a_1, ..., a_k, x_1, ..., x_n) \leftrightarrow \phi_i(a_1, ..., a_k, x_{2n+1}, ..., x_{3n})$

(c) $\neg\phi_i(a_1, ..., a_k, x_{n+1}, ..., x_{2n}) \leftrightarrow \phi_i(a_1, ..., a_k, x_{2n+1}, ..., x_{3n})$

But the conjunction of these conditions implies a contradiction. Hence $i = 0$

Take now the last n-elements of Z, say $z_1 < z_2 < ... < z_n$. Let therefore $I = Z \setminus \{z_1, ..., z_n\}$. Taking two sequences $x_0 < x_1 < ... < x_n$ and $x_0 < y_1 < ... < y_n$ of elements in I, for $a_1, ..., a_k < x_0$ we will have:

(a) $\phi_i(a_1, ..., a_k, x_1, ..., x_n) \leftrightarrow \phi_i(a_1, ..., a_k, z_1, ..., z_n)$

(b) $\phi_i(a_1, ..., a_k, y_1, ..., y_n) \leftrightarrow \phi_i(a_1, ..., a_k, z_1, ..., z_n)$, from which follows

(c) $\phi_i(a_1, ..., a_k, x_1, ..., x_n) \leftrightarrow \phi_i(a_1, ..., a_k, y_1, ..., y_n)$

Namely, I is *a set of indiscernibles*. QED

The Main Lemma gives us indiscernibles for a finite Γ; at the opposite, in the following we need it for the *infinite* class Δ_0. Notice that the *Main Lemma* is formalizable in any theory (say PA) which has a truth predicate for the formulas concerned. Let therefore $\mathcal{M}$ a non standard model of PA and suppose that Kanamori-McAloon holds in it. Since the lemma of existence of indiscernibles holds in it for all standard numbers e, m, n, by overspill holds for some non-standard c and for what "look" to $\mathcal{M}$ like the first c formulas from Δ_0, having $2c$ variables and therefore we note that holds for all infinite Δ_0 standard formulas.

Recall that a general theorem of *absoluteness* holds.

Theorem 101. *Let $\mathcal{M} \models$ PA and let $J \subseteq \mathcal{M}$ an initial segment of it (i.e. if $a \in \mathcal{M}$, $b \in J$ and $a < b$, then $a \in J$). Let $\phi(x) \in \Delta_0$ and let $c \in J$. Then $\mathcal{M} \models \phi(c)$ if and only if $J \models \phi(c)$.*

Let us consider now $\Gamma = \Delta_0$. We show how to use indiscernibles to find initial segments that are models of PA.

Theorem 102. *Let $\mathcal{M}$ be a model of PA and let $x_1 < x_2 < x_3 < ...$ indiscernibles for the class Δ_0 of formulas. Let $J = \{y \in M | \exists i(y < x_i)\}$. Then also J is a model of PA.*

Proof. First we prove the closure under the operations:

(a) suppose that $i < j < k < e$ and $a < x_i$; if $a + x_j \geq x_k$, then there exists $b \leq a(b + x_j = x_k)$; but from indiscernibility also we have $b + x_j = x_e$, from which $x_k = x_e$, against the hypothesis that $x_k < x_e$. Hence $a + x_j < x_k$ and since x_k is un indiscernible also we have $a + x_j \in J$. In particular $x_i + x_j \leq x_k$. It follows that J is closed under addition.

(b) Under the same conditions we also have $a \cdot x_j < x_k$. Otherwise we would have, for some minimal $a = b + 1$, that $bx_j < x_k \leq (b + 1)x_j$. By indiscernibility, also $x_e \leq (b + 1)x_j$. Moreover, $(b + 1)x_j = bx_j + x_j < x_k + x_j$. But from the previous point $x_k + x_j \leq x_e$, from which $x_e \leq (b + 1)x_j < x_e$ (contradiction). Hence $ax_j < x_k$. It follows the closure under multiplication.

Now we show the closure under induction. We remark therefore that truth of a formula in J can be reduced to truth of a Δ_0 formula in $\mathcal{M}$. For instance, let us consider $\exists v_1 \forall v_2 \exists v_3 \psi(w, v_1, v_2, v_3)$. Let $a < x_i$. But $J \models \exists v_1 \forall v_2 \exists v_3 \psi(a, v_1, v_2, v_3)$ iff there exists $i_1 > i$ such that for all $i_2 > i_1$, exists $i_3 > i_2$, such that:

$$J \models \exists v_1 < x_{i_1} \forall v_2 < x_{i_2} \exists v_3 < x_{i_3} \psi(a, v_1, v_2, v_3)$$

But truth of Δ_0 formulas is preserved in extensions of models and then these remains true in $\mathcal{M}$. Being the x_j indiscernibles, the above formulas can be reduced to a unique formula, i.e. it will be true in $\mathcal{M}$ that $\exists v_1 < x_{i+1} \forall v_2 < x_{i+2} \exists v_3 < x_{i+3} \psi(a, v_1, v_2, v_3)$.

Hence $J \models \exists v_1 \forall v_2 \exists v_3 \psi(a, v_1, v_2, v_3)$ if and only if:

$$M \models \exists v_1 < x_{i+1} \forall v_2 < x_{i+2} \exists v_3 < x_{i+3} \psi(a, v_1, v_2, v_3)$$

But in $\mathcal{M}$ will be true the least number principle (equivalent to the induction). Hence there is a *minimum* $a^* < x_i$ such that the formula holds in the model, i.e.:

$$M \models \exists v_1 < x_{i+1} \forall v_2 < x_{i+2} \exists v_3 < x_{i+3} \psi(a^*, v_1, v_2, v_3)$$

but from this follows $J \models \exists v_1 \forall v_2 \exists v_3 \psi(a^*, v_1, v_2)$, and the the least number principle also holds in J. QED

Independence of the Kanamori-McAloon principle. Let $\mathcal{M}$ a non-standard model of PA and let c a non-standard element.

(A) Recall that PA proves the finite "Ramsey". Hence take the *minimum* w such that in $\mathcal{M}$ is true that $w \longrightarrow (3c+1)^{2c+1}_c$, namely, that for all $f : [w]^{2c+1} \to c$, there is Y of cardinality bigger than $3c+1$ homogeneous.

(B) Let us suppose, by contradiction, that PA proves Kanamori-McAloon's principle. Therefore the above $\mathcal{M}$ is a nonstandard model of Peano Arithmetic where this principle holds. Given the above w, let therefore d the *minimum* such that, if $f_0, ..., f_c : [d]^{2c+1} \longrightarrow d$ are regressive, then exists $Y \subseteq [c, d)$ such that $|Y| \geq w$ and Y is min-homogeneous for $f_0, ..., f_c$.

Following the steps of the proof of the *Main Lemma* inside the model, we can indeed obtain a set $c \leq I < d$ of cardinality bigger or equal than c, that according to $\mathcal{M}$ is a set of indiscernibles for Δ_0-formulas coded with code at most c (and then in particular *all* Δ_0-formulas coded by standard numbers). Recall that in PA we can define *partial* truth predicates, in particular for the Δ_0 formulas. This, toghether with the usual coding apparatus, is all what is needed to formalize the proof of the result about the existence of indiscernibles and rebuild it inside the model $\mathcal{M}$. Following the steps of the proof of the *Main Lemma* inside the model, we can obtain a set $I \subseteq Y$ of cardinality bigger than c (where $Y \subseteq [c, d)$ is the above mentioned set), that according to $\mathcal{M}$ is a set of indiscernibles for Δ_0-formulas coded with code at most c (and then in particular *all* Δ_0-formulas coded by standard numbers).

Let therefore $x_0 < x_1 < x_2 < ...$ an initial segment of I and let $J = \{y \in M | y < x_i, for\ some\ i\}$. For the previous theorem J is a model of PA. Moreover $c \in J$, but $d \notin J$. Note that:

(A') For finite Ramsey's theorem (true in J) there is $v \in J$ such that in J is true (A), i.e. for all functions $f : [v]^{2c+1} \longrightarrow c$ there exists Z homogeneous of cardinality bigger or equal to $3c+1$. Since all functions and sets needed for this statement are included in J, this holds also in $\mathcal{M}$. But once fixed $c, 2c$, and $3c+1$, according to (A) the set w was minimal in $\mathcal{M}$ to satisfy this statement, hence $w \leq v$. Therefore $w \in J$.

(B') Analogously, if $h \in J$ and is true in J that if the $f_0, ..., f_c : [h]^{2c+1} \longrightarrow h$ are regressive, by the hypothesis that the Kanamori-McAloon's principle is true in models of PA, there exists Y min-homogeneous for them of cardinality greater or equal to w, and arguing as above, this is also true for $\mathcal{M}$. But according to (B) d was the minimum for which this is true in $\mathcal{M}$. Hence $d \leq h$ and therefore $d \in J$. But we also had $d \notin J$ (contradiction)

It follows that Peano Arithmetic does not prove Kanamori-McAloon statement and consequently does not prove the Paris-Harrington either.

6.3. The Hydra game

Kirby and Paris (1982) proved an extension of Goodstein's original result, mentioned in the introduction: recall that each integer can be written in base $b \geq 2$, in form $b^{n_0} \cdot c_0 + ... + b^{n_k} \cdot c_k$ where $n_0 \geq n_1 \geq ... \geq n_k$. The expontents in turn will be written in base b. For example, if $n = 266$ and $b = 2$, we can write n in base b as:

$$n = 2^{2^{2+1}} + 2^{2+1} + 2^1$$

Let us define $G_n(x)$ by cases as follows: $G_n(m) =$ "the number obtained replacing each n in base n representation of m, with $n+1$, if $m \neq 0$ ($G_n(m) = 0$ otherwise)".

For instance $G_2(266) = 3^{3^{3+1}} + 3^{3+1} + 3^1$; a Goodstein's sequence $n_0, n_1, n_2...$ is such that $n_0 = n$ and $n_{k+1} = G_{k+2}(n_k) - 1$ if $n_k > 0$; in our previous element:

$$\begin{aligned}
&n_0 = n = 266\\
&n_1 = G_2(n_0) - 1 = 3^{3^{3+1}} + 3^{3+1} + 2\\
&n_2 = G_3(n_1) - 1 = 4^{4^{4+1}} + 4^{4+1} + 1\\
&n_3 = G_4(n_2) - 1 = 5^{5^{5+1}} + 5^{5+1}\\
&\vdots
\end{aligned}$$

This is called "the Goodstein sequence for m starting at 2". This definition can be generalized to "the Goodstein sequence for m starting at r", for each $r \geq 2$. The *Goodstein's theorem* (1944) says that each such sequence converges to 0, i.e. for all numbers n, there is a number k such that $n_k = 0$. However this k is of very big dimension, for instance, if $n = 4$, then $k = 3 \cdot 2^{40265321} - 3$.

Well, although the Goodstein's theorem is formalizable in the language of Peano arithmetic at first order PA, this theory (which we assume to be consistent) is not able to prove the corresponding formal statement. It is in fact possible to prove that the statement of the Goodstein's theorem is equivalent to the provability in PA, of the principle $TI(\varepsilon_0)$ of transfinite induction up to ε_0, but it is well known by Gentzen's historical results that this principle is not provable in PA and in fact, it is the principle used by Gentzen to show the consistency of this theory. The game of Hercules versus Hydra of Kirby and Paris is a combinatorial game whose termination is true, although this truth is not provable in Peano Arithmetic. The Hydra is the monster of Graeco-Roman mythology with many heads, which grow multiplying every time they are cut off. Hercules' aim is to cut off all Hydra's heads. Actually, in the game hydras are graphs, in particular they are finite rooted trees, i.e. connected graphs with no cycles and a specific node, the root. A head is an edge (a, b) where b is a leaf of the tree and the node a is its (unique) predecessor, that we call "its father" (sometimes a is also called "neck"). After Hercules cuts off a head, the hydra grows new heads, namely the tree changes according to this rule:

(a) At stage $n \geq 1$ Hercules chop one head, namely an edge (a, b) as above, where b is a leaf.

(b) At this point the Hydra grows n new heads: from the predecessor of a, say s (sometimes called "grandfather of b", or "trunk"), the monster generates n copies rooted in s of what remains of the subtree generated by a after the decapitation, i.e. after removing (a, b).

(c) if one of the two nodes of the edge just chopped off coincides with the root, no new head is grown.

(d) Hercules wins the battle if he reduces in a finite number of moves the monster to its root.

Actually Hercules always kill the Hydra: we know that every recursive strategy is a winning strategy, however, the statement "Every recursive strategy is a winning strategy", although true, is not provable in Peano Arithmetic. Here we illustrate the original proof due to Kirby

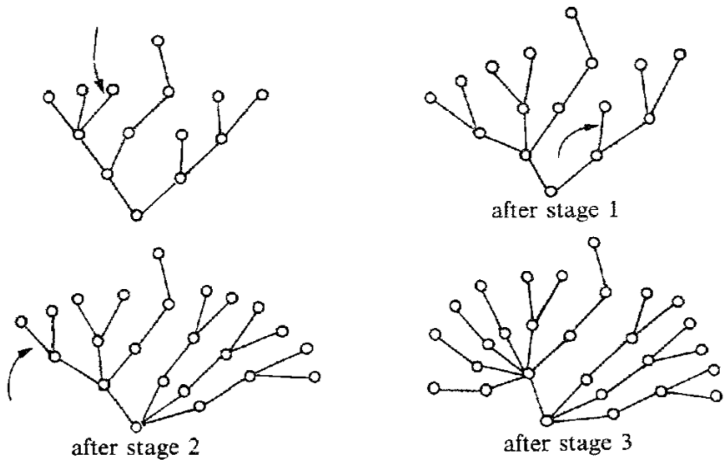

Figure 7. Reproduction scheme of the Hydra, from the paper by Kirby and Paris.

and Paris, which makes use of the method of *indicators* and of the theory of α-*large sets* (where α is an ordinal) introduced by Ketonen and Solovay. The *indicators* method comes from model theory, was introduced by Kirby and Paris in the 1970s and was later used to find proofs of independence. Other, proof theoretic, rather than model-theoretic methods were used for the same purpose. For instance, Carlucci (2003) followed another route, using the sequents calculus and showing that a relation holds between Kirby-Paris *Hydra Game* and Gentzen Reduction Strategy by a natural interpretation of derivations as hydras.

Definition 48. *Suppose that Θ is a set of cuts of a countable non-standard model $\mathcal{M} \models I\Sigma_1$. A Σ_1 formula $Y(x, y, z)$ is called an indicator for Θ in $\mathcal{M}$, if the following hold:*

(a) $\mathcal{M} \models \forall x \forall y \exists! z Y(x, y, z)$; *in other words, this formula defines a function in the model $\mathcal{M}$ (therefore we can write $Y^{\mathcal{M}}(x, y) = z$ in place of $\mathcal{M} \models Y(x, y, z)$).*

(b) For all $a, b \in \mathcal{M}$, $Y^{\mathcal{M}}(a, b) > \mathbb{N}$ (meaning that is bigger of all natural numbers) if and only if there exists $I \in \Theta$ of $\mathcal{M}$, such that $a \in I < b$ (which means that contains a and all its elements are less than b). It is actually provable that there are $2^{\aleph_0}$ such initial segments.

(c) For a, b, c, d elements of $\mathcal{M}$, if $a \leq b$ and $c \leq d$, then $Y^{\mathcal{M}}(b, c) \leq Y^{\mathcal{M}}(a, d)$.

See Paris (1980) and Hájek and Pudlák (1993), pp. 245-260 for a detailed discussion of the subject. For example, using the notation from the previous section, this is an indicator for $\Theta = \{I \subseteq_e \mathcal{M} | I \models \mathsf{PA}\}$ where $\mathcal{M} \models I\Sigma_1$ (we say that it is *an indicator for models of* PA *in* $I\Sigma_1$):

$$Y(a, b) = max\ c\ such\ that\ [a, b] \rightarrow_* (c + 1)^c_c$$

The following theorem shows some important properties of indicators for $\Theta = \{I \subseteq_e \mathcal{M} | I \models \mathsf{T}\}$. Actually, this is the case in which we are most interested and we speak in that case simply of *indicators for models of* T.

Theorem 103. *Let* T *be a recursive extension of $I\Sigma_1$ and $Y(x, y) = z$ an indicator for $\Theta = \{I \subseteq_e \mathcal{M} | I \models \mathsf{T}\}$ in any countable model $\mathcal{M}$ of* T. *Hence the following hold:*

(a) $\mathsf{T} \nvdash \forall x \forall z \exists y Y(x, y) \geq z$

(b) *for all natural number n,* $\mathsf{T} \vdash \forall x \exists y Y(x,y) \geq \overline{n}$

(c) $\mathbb{N} \models \forall x \forall z \exists y Y(x,y) \geq z$

(d) *For any provably total function $f(x)$ there exists a natural number n such that:*

$$\mathsf{T} \vdash \forall x (f(x) < g_n())$$

where $g_n(x) =$ least y such that $Y(x,y) \geq n$ (we also say that g_n forms an envelope for T-provably total funcions).

The next tools we need come from the theory of $\alpha-$largeness and of fundamental sequences of ordinals, i.e. of strictly increasing sequence of ordinals whose supremum is a limit ordinal. First of all, we must therefore recall some important notions, such as that of *Cantor's normal form*, relative to the ordinals less than ε_0, i.e. the fact that each ordinal can be written as a finite sum as follows:

(a) 0 is an ordinal.

(b) if $\alpha_0, ..., \alpha_n$ are ordinals and $\alpha_0 \geq ... \geq \alpha_n$, then $\omega^{\alpha_0} + ... + \omega^{\alpha_n}$ is an ordinal (where $\omega^1 = \omega$ and $\omega^0 = 1$ and the ordinal is limit in case $\alpha_n \neq 0$).

(c) $\omega^{\alpha_0} + ... + \omega^{\alpha_n} \geq \omega^{\beta_0} + ... + \omega^{\beta_k}$ if and only if either for some i, $\alpha_i \geq \beta_i$ and for all $j < i, \alpha_j = \beta_j$, or $n > k$ and for all $i \leq k, \alpha_i = \beta_i$.

Then we define the following sort of "predecessor" operation $\{\alpha\}(k)$, due to Ketonen and Solovay.

Definition 49. *Let $\alpha < \varepsilon_0$ and $k < \omega$. Then:*

(a) $\{0\}(k) = 0$

(b) *If $\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_n}$ and $\alpha_n = 0$ (i.e. α is a successor), then:*

$$\{\alpha\}(k) = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}}$$

(c) *If $\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_n}$ and $\alpha_n = \beta + 1$ for some β, then*

$$\{\alpha\}(k) = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}} + \omega \cdot k$$

(d) *If $\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_n}$ and α_n is limit, then:*

$$\{\alpha\}(k) = \omega^{\alpha_0} + ... + \omega^{\alpha_{n-1}} + \omega^{\{\alpha_n\}(k)}$$

This operation gives rise to a sequence:

$$\{\alpha\}(0) < \{\alpha\}(1) < \{\alpha\}(2) < ... < \alpha$$

where $\alpha = sup_{n \in \omega}\{\alpha\}(n)$. Let us see some examples to make this definition intuitively clearer:

(a) $\{\omega\}(k) = \{\omega^1\}(k) = \{\omega^{0+1}\}(k) = \omega^0 \cdot k = k$.

(b) $\{\omega^\omega\}(k) = \omega^{\{\omega\}(k)} = \omega^k$.

(c) $\{\omega^{n+1}\}(k) = \omega^n \cdot k$.

With the writing $\{\alpha\}(n_0, n_1, ..., n_k)$ we abbreviate the iteration:

$$\{...\{\{\alpha\}(n_0)\}(n_1)...\}(n_k)$$

Lastly, we introduce the notion of "$\alpha-$largeness", which generalises the notion of "largeness"that we encountered in the Paris and Harrington theorem. Let $X = n_0 < n_1 < ... < n_k$ be a finite set of numbers. We say that:

(a) X is $1-$large if and only if it contains at least two elements.

(b) X is $\alpha-$large if and only if $X - \{n_0\}$ is $\{\alpha\}(n_1)$-large.

Arguing by induction on α, we see that actually this condition is equivalent to say that $\{\alpha\}(n_1, n_2, ..., n_k) = 0$. Indeed X is $\alpha-$large if and only if $X - \{n_0\}$ is $\{\alpha\}(n_1)-$large, if and only if by induction hypothesis (since $\{\alpha\}(n_1) < \alpha$):

$$\{\{\alpha\}(n_1)\}(n_2, ..., n_k) = 0$$

that is, $\{\alpha\}(n_1, n_2, ..., n_k) = 0$.

For our purposes, it is important to emphasise this result of Ketonen and Solovay (1981).

Theorem 104. *The following is an indicator for models of Peano Arithmetic* PA*:*

$$Y(a, b) = \textit{greatest } c \textit{ such that the interval } [a, b] \textit{ is } \omega_c - \textit{large}$$

where $\omega_0 = \omega$ *and* $\omega_{n+1} = \omega^{\omega_n}$*, although this statement is independent of* PA*:*

$$\forall a \forall c \exists b([a, b] \textit{ is } \omega_c - \textit{large})$$

To continue this exposition, it is necessary to further examine the mechanism of the so-called *fundamental sequences* of ordinals. Hence let us write $\beta \rightarrow_n \alpha$ to mean that, for $j_0, ..., j_k \leq n$, we have $\alpha = \{\beta\}(j_0, ..., j_k)$. We will write $\beta \Rightarrow_n \alpha$ in case $j_0 = j_1 = ... = n$.

Lemma 34. *The following basic properties hold:*

(a) *if* $\beta \Rightarrow_n \alpha$*, for* $n > 0$*, then* $\omega^\beta \Rightarrow_n \omega^\alpha$*.*

(b) *If* $0 < i < j \leq n$*, then* $\{\beta\}(j) \Rightarrow_n \{\beta\}(i)$*.*

(c) *Let us write* $\alpha \gg \beta$ *to mean that* $\alpha = \omega^{\gamma_0} + ... + \omega^{\gamma_m}$ *and* $\beta = \omega^{\delta_0} + ... + \omega^{\delta_k}$ *where* $\gamma_0 > \gamma_1 > ... > \gamma_m \geq \delta_0 > \delta_1 > ... > \delta_k$*. Hence if* $\beta > 0$ *and* $\alpha \gg \beta$*, then* $\{\alpha + \beta\}(n) = \alpha + \{\beta\}(n)$*.*

(d) *If* $\alpha \gg \beta$ *and* $\beta \Rightarrow_n \gamma$*, then* $\alpha + \beta \Rightarrow_n \alpha + \gamma$*.*

(e) *If* $\alpha < \varepsilon_0$ *and* $n \geq 0$*, then* $\alpha \Rightarrow_n 0$*.*

(f) $\beta \Rightarrow_n \alpha$ *if and only if* $\beta \rightarrow_n \alpha$*.*

(g) *If* $\beta \rightarrow_n \alpha$ *and* $0 < n \leq n_0 < ... < n_k$*, then* $\{\beta\}(n_0, ..., n_k) \geq \{\alpha\}(n_0, ..., n_k)$*.*

Proof. We illustrate only a sketch of the proofs of the more complex points:

(a) by induction on β. We can consider just the case $\alpha = \{\beta\}(n)$, since the general case follows arguing once again inductively.

   i. If $\beta = 0$, this is immediate.

ii. If β is limit, then notice that $\{\omega^\beta\}(n) = \omega^{\{\beta\}(n)} = \omega^\alpha$.

iii. If β is a successor, and then $\beta = \alpha + 1$, observe that for $k < l < \omega$, we have $\omega^\beta \cdot l \Rightarrow_n \omega^\beta \cdot k$. Indeed, by point 5. $\omega^\beta \cdot (l-k) \Rightarrow_n 0$; hence $\omega^\beta \cdot l = \omega^\beta \cdot k + \omega^\beta \cdot (l-k) \Rightarrow_n \omega^\beta \cdot k$ by point 4. Lastly:

$$\omega^\beta = \omega^{\alpha+1} \Rightarrow_n \{\omega^{\alpha+1}\}(n) = \omega^\alpha \cdot n \Rightarrow_n \omega^\alpha$$

(b) By transfinite induction, using points 1. and 4.

(c) It follows from the fact that, under the hypotesis of the lemma, the sum $\alpha + \beta$ is just the concatenation of the respective normal forms. Hence, look at the index δ_k.

(d) This is an application of the previous point: notice that if $\gamma = \{\beta\}(n)$, then $\{\alpha+\beta\}(n) \Rightarrow_n \alpha + \{\beta\}(n) = \alpha + \gamma$, hence $\alpha + \beta \Rightarrow_n \alpha + \gamma$.

(e) By induction on α:

i. $\alpha = 0$, immediate.

ii. If $\alpha = \gamma + 1$, then $\{\gamma + 1\}(n) = \gamma$, namely $\gamma + 1 \Rightarrow_n \gamma$. But by the inductive hypothesis $\gamma \Rightarrow_n 0$.

iii. If α is limit, say in Cantor normal form $\alpha = \omega^{\gamma_0} + ... + \omega^{\gamma_m}$; being limit, $\gamma_m \neq 0$. Suppose $\gamma_m = \sigma + 1$; hence:

$$\{\omega^{\gamma_0} + ... + \omega^{\gamma_m}\}(n) = \omega^{\gamma_0} + ... + \omega^{\gamma_{m-1}} + \{\omega^{\delta+1}\}(n)$$

Now apply twice the inductive hypothesis and point 4. The case of γ_m limit is analogous.

(f) Follows from point 2. and is left as an exercise.

(g) Induction on β. If $\beta = 0$ it is clear. Otherwise, assume the result holds below β and observe that if $\beta \rightarrow_n \alpha$ and $0 < n \leq n_0 < ... < n_k$, then $\beta \rightarrow_{n_0} \alpha \rightarrow_{n_0} \{\alpha\}(n_0)$, from which follows $\{\beta\}(n_0) \rightarrow_{n_0} \{\alpha\}(n_0)$. By inductive hypothesis:

$$\{\{\beta\}(n_0)\}(n_1, ..., n_k) \rightarrow_{n_0} \{\{\alpha\}(n_0)\}(n_1, ..., n_k)$$

which is our desired result.

QED

The next operator we are going to introduce is another "predecessor" function (sometimes called *Goodstein predecessor*) denoted $\langle\alpha\rangle(n)$. It defines a sequence of ordinals converging to α, but faster than $\{\alpha\}(n)$, in the sense that:

$$\langle\alpha\rangle(n) \rightarrow_n \{\alpha\}(n)$$

Definition 50. *Let $\alpha < \varepsilon_0$. Then:*

(a) $\langle 0\rangle(n) = 0$.

(b) $\langle\alpha + 1\rangle(n) = \alpha$.

(c) $\langle\omega^\delta(\alpha + 1)\rangle(n) = \omega^\delta \cdot \alpha + \omega^{\langle\delta\rangle(n)} + \langle\omega^{\langle\delta\rangle(n)}\rangle(n)$.

For instance, $\langle 3\rangle(n) = 2$ and $\langle \omega^3(\alpha+1)\rangle(n) = \omega^3 \cdot \alpha + \omega^2 \cdot n + \omega$.

Moreover, with the notation $f_{m,n}(x)$ we will denote the function that takes the representation of m in base n and replaces n with x. In other words, if:

$$n = 2^{2^{2+1}} + 2^{2+1} + 2^1$$

and $x = n + 1$, then:

$$f_{m,n}(x) = 3^{3^{3+1}} + 3^{3+1} + 3^1$$

and in general, if $m = a_0 \cdot n^0 + a_1 \cdot n^1 + ... + a_k \cdot n^k$ then:

$$f_{m,n}(x) = a_0 \cdot x^{f_{0,n}(x)} + a_1 \cdot x^{f_{1,n}(x)} + ... + a_k \cdot x^{f_{k,n}(x)}$$

In terms of the previous notation, for $m > 0$, $G_n(m) = f^{m,n}(n+1) - 1$; we call instead $o_m(n)$ the number $f_{m,n}(\omega)$. These notions are related as follows.

Theorem 105. *(a) for $m \geq 0$, $n > 1$, if $o_{n+1}(m) = \alpha$, then $o_{n+1}(m \dot{-} 1) = \langle\alpha\rangle(n)$.*

(b) For $n > 1$, $\langle o_n(m)\rangle(n) = o_{n+1}(G_n(m))$.

Proof. (a) If $m = 0$ this is obviuos, so let us consider the base $n+1$ representation of $m > 0$:

$$m = \sum_{i=0}^{p} a_i (n+1)^{f_{i,n+1}}$$

where $a_i \leq n$. Let j be the minimum index such that $a_j \neq 0$. Notice that if $j = 0$ the result is immediate, hence let us assume thsat $j > 0$ and assume by inductive hypothesis that the result holds for al $s < m$. Observe that $o_{n+1}(m \dot{-} 1)$ is the sum of the following addends:

(i) $(\sum_{i=j+i}^{p} \omega^{f_{i,n+1}(\omega)} a_i) + \omega^{f_{j,n+1}(\omega)}(a_j - 1)$

(ii) $o_{n+1}(n \cdot (n+1)^{f_{j,n+1}(n+1)-1})$

(iii) $o_{n+1}((n+1)^{f_{j,n+1}(n+1)-1} - 1)$

On the other hands, $\langle\alpha\rangle(n)$ is the sum of (i) plus:

(iv) $\omega^{\langle f_{j,n+1}(\omega)\rangle(n)} \cdot n$

(v) $\langle \omega^{\langle f_{j,n+1}(\omega)\rangle(n)}\rangle(n)$

We see that actually (iii)=(v) and (ii)=(iv):

(a) (iii)=(v). By induction hypothesis (iii) is equal to $\langle o_{n+1}((n+1)^{f_{j,n+1}(n+1)-1})\rangle(n)$ and this is (v) by definition.

(b) (ii)=(iv). By definition (ii) is $\omega^{f_{j,n+1}(\omega)-1} \cdot n$, namely to $\omega^{o_{n+1}(f_{j,n+1}(n+1)-1}$. Still by the inductive hypothesis this is equal to (iv).

(b) The proof is along the same lines and we omit it.

QED

By using this notation, if for instance $n_0, n_1, n_2...$ is the above mentioned Goodstein sequence, then it correspond to a sequence of ordinals in Cantor normal form:

$$o_n(n_0), o_{n+1}(n_1), o_{n+2}(n_2)...$$

where $o_n(n_0) = \omega^{\omega^{\omega+1}} + \omega^{\omega+1} + \omega$, that is:

$$\omega^{\omega^{\omega+1}} + \omega^{\omega+1} + \omega, \omega^{\omega^{\omega+1}} + \omega^{\omega+1} + 2, \omega^{\omega^{\omega+1}} + \omega^{\omega+1} + 1...$$

that can also be written as:
$\omega^{\omega^{\omega+1}} + \omega^{\omega+1} + \omega$, $\langle\omega^{\omega^{\omega+1}} + \omega^{\omega+1} + \omega\rangle(n)$, $\langle\omega^{\omega^{\omega+1}} + \omega^{\omega+1} + \omega\rangle(n, n+1)$, $\langle\omega^{\omega^{\omega+1}} + \omega^{\omega+1} + \omega\rangle(n, n+1, n+2)$...

We therefore come to a central result for our purpose.

Theorem 106. *Let $n_0, n_1, n_2....$ a Goodstein sequence for m starting at n and let k be the minimum such that $n_k = 0$. Then the interval $[n-1, n+k-1]$ is $o_n(m)$-large.*

Proof. Let us consider the cooresponding sequence of ordinals:

$$o_n(m) = o_n(n_0) = \alpha$$
$$o_{n+i}(n_i) = \langle\alpha\rangle(n, n+1, .., n+i-1)$$
$$\vdots$$
$$o_{n+k}(n_k) = \langle\alpha\rangle(n, n+1, .., n+k-1) = 0$$

From the relationships previously shown between the two predecessor operators and the result on p.166, point 6., it can be seen that:

$$\begin{aligned} &\{\alpha\}(n, n+1, ..., n+k-1) \leq \\ &\leq \{\langle\alpha\rangle(n)\}(n+1, .., n+k-1) \\ &\leq \{\langle\alpha\rangle(n, n+1)\}(n+2, .., n+k-1) \\ &\vdots \\ &\leq \langle\alpha\rangle(n, n+1, ..., m+k-1) = 0 \end{aligned}$$

and therefore $[n-1, n+k-1]$ is α-large. QED

We remark that this proof *can be formalised and carried out in* PA. We now want to apply this result to show that the statement (true and formalisable in the language of arithmetic):

> (*) "for each m and each n, the Goodstein sequence for m starting at n eventually hits zero".

is not provable in PA. Suppose by contradiction it is actually provable. As a consequence of the independence result of Ketonen and Solovay (theorem 3) mentioned at the beginning of the paragraph and some mathematics for the 'indicators', we can find a nonstandard model $\mathcal{M}$ of PA and a non standard element c of this model in which the statement:

> (**) "there exists an element b such that the interval $[1, b]$ is ω_c-large".

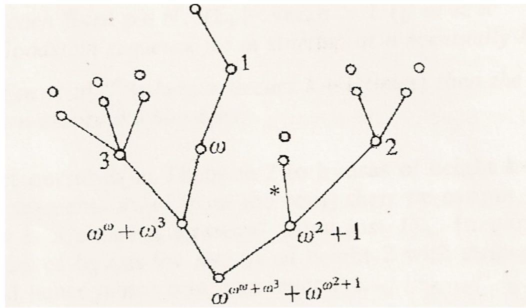

Figure 8. Assignment of ordinals to the Hydra (from Kirby and Paris paper).

is false. Hence take $d = 2^{2^{2}}$ with c iteration of the exponentiation, so that $o_2(d)$ is just ω_c. By (*), that, as a theorem, is true in all models, we can take an element e of the model $\mathcal{M}$ such that $d_e = 0$. By the theorem 106 and the subsequent remark, we have in $\mathcal{M}$ that $[1, 2+e-1]$ is ω_c large, but this contradicts (**). Hence (*) is not provable in PA.

Let us finally return to our Hydra and show that the true statement:

(***) "Every recursive strategy for Hercules is a winning strategy"

is not provable in PA. We begin by assigning the Hydra nodes an ordinal less than ε_0 according to this criterion:

(a) To the leafs, assign 0.

(b) To each other node assign $\omega^{\alpha_0} + ... + \omega^{\alpha_n}$ where $\alpha_0 \geq ... \geq \alpha_n$ are the ordinals assigned to the sons of that node. The ordinal of the Hydra is the ordinal assigned to the root.

We now consider the strategy τ based on this algorithm: starting from the root, go to the son (i.e. the immediate successor) labelled with the smallest ordinal among all the sons; continue in the next nodes in the direction of the leaves, always moving towards the immediate successor labelled with the smallest ordinal among the immediate successors, following this criterion until you reach a head. When you reach it, cut it off.

If we denote $\tau(\alpha, n)$ the operator that following the strategy τ, maps the ordinal α at the root of the hydra after the stage $n-1$, to the ordinal at the root of the Hydra after the stage n, then it is clear that $\tau(\alpha, n) < \alpha$ and therefore Hercules eventually wins.

However, it occurs that this fact cannot be proved in PA. It is in fact the case that:

$$\tau(\alpha, n) = \{\alpha\}(n+1)$$

Hence a proof of the unprovability of (***) follows the line of the analogous proof for (*). Actually the proof that τ is a winning strategy is equivalent to the ε_0-induction, with $\{\alpha\}(n)$ as predecessor function, that in turn is equivalent to the fact (consequently unprovable in PA) that the function:

$$\lambda x \lambda v. g_v(x) = min.y \geq x \; such \; that \; [x, y] \; is \; \omega_v - large$$

is provably total in PA. A proof-theoretic method for proving incompleteness of PA is indeed based on the classification of provably total functions in PA (see Schwichtenberg, Wainer (2011)).

The proof we have illustrated does indeed appear rather complex, which is why it was felt necessary to find alternative proofs. With this goal, a proof of Goodstein's theorem using

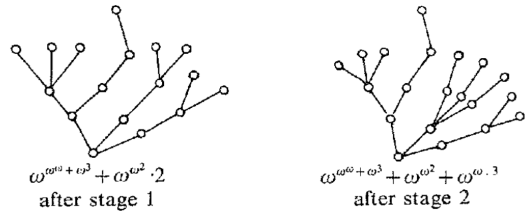

Figure 9. Assignment of ordinals to the Hydra (from the Kirby and Paris paper)

methods from computability theory, rather than model theory, was given for instance in Cichon (1983), which connects the problem of the provability of Goodstein's theorem to well-known results on the recursion theoretic hierarchies of functions, showing that if the function:

$$G(a) = least\ k\ such\ that\ a_k = 0$$

where $a = a_0, a_1.a_2....$ is a Goodstein sequence, were provably total in PA, then so would be the Hardy function H_{ε_0}. Recall that the Hardy hierarchy $\{H_\alpha\}_{\alpha \leq \varepsilon_0}$ is defined as follows:

(a) $H_0(n) = n$

(b) $H_{\alpha+1}(n) = H_\alpha(n+1)$

(c) $H_\lambda(n) = H_{\{\lambda\}(n)}(n)$ for λ limit.

Adding the clauses $\{\varepsilon_0\}(0) = \omega$ and $\{\varepsilon_0\}(n+1) = \omega^{\{\varepsilon_0\}(n)}$ to our previous definition of this operator. Now, it is provable that H_{ε_0} majorizes all functions provably total in PA but it is not itself provably total in this theory. The relationship that exists between the study of the so-called "fast growing functions"and the themes of this chapter has been the subject of extensive in-depth analysis in Buchholz and Wainer (1987) and Ketonen and Solovay (1981).

Many scientific contribution which is worth mentioning are indeed related to the ordinal analysis of theories and its connection to provably total functions. Actually these proof-theoretic works are very difficult and would require further mathematical premises such that their detailed analysis go beyond the scope of this book.

As we have repeatedly recalled in this book, Kreisel (1952) showed that the functions provably total (called also *provably recursive* or *provably computable*) in Peano Arithmetic are those definable by recursion over well-ordering of order-type less than ε_0. At the origin of everything there are Gentzen's 1934 - 1939 consistency proofs for PA, obtained by adding the transfinite induction principle up to ε_0 for primitive recursive (more precisely, the *elementary computable*) predicates to an acceptable finitistic basis, so that we say that the proof-theoretic ordinal of Peano Arithmetic is ε_0.

6.4. Further developments and guide for further study

Beklemishev (2006) proposed a variation of the Hydra game, called 'Worm battle' (see Carlucci (2005) to understand the connection with other works mentioned here). A system of ordinal notation emerges amazingly in this context from a certain system of propositional modal logic. Beklemishev's very original work proposes an algebraic approach to proof-theoretic analysis based on the notion of graded provability algebra, that is, Lindenbaum boolean algebra of a theory enriched by additional modal operators. Another modified version of the Hydra Game is due to Buchholz (1987), that extended Kirby-Paris' Hydra Game by defining a game on labeled finite trees (following a suggestion from Martin Gardner) in which a hydra grew not only in width but also in height and that is independent of a certain strong subsystem of analysis. A proof of the independence result of a restricted Buchholz style-Hydra Game of certain subsystems of analysis is given also in Hamano and Okada (1998). Both, the Kirby-Paris Hydra as well as the Buchholz type of Hydras have been studied in the context of one of the most important research topic within Proof Theory, namely *ordinal analysis*, which started with Gentzen's often mentioned work on the consistency of arithmetic. In Proof Theory, the so-called "ordinal analysis" assigns tranfinite ordinals to mathematical theories as a measure of their consistency strength or computational power (see Wolfram Pohlers (1993), Rathjen (2006) and Arai (2020)). The proof-theoretic ordinal of a theory can be also defined as the smallest ordinal that this theory cannot prove to be well-founded and can be seen also as a measure of the system's ability to prove the totality of computable functions.

If we say that a Hydra has ordinal strength α if a proof of its termination requires a theory with ordinal strength at least α, then in in this sense, the Kirby-Paris Hydra, has ordinal strength ε_0, while the ordinal strength of the Buchholz Hydra exceeds even all the ordinals expressible by the so-called Buchholz's ψ ordinal functions (see Endrullis, Klop and Overbeek (2021)).

A remarkable characteristic of Carlucci (2003) proof-theoretic approach to the Hydra Game is that it uses a very natural interpretation of the derivations in a system of Peano Arithmetic as hydras.

The results of this work are parallel and, although independent, somewhat related to the work of Hamano and Okada (1998), but the author emphasises that the common 'diagrammatical' flavour of the idea of the proofs is made much more evident in his own approach. In particular, no mention of ordinals nor of transfinite hierarchies is made. Carlucci is able to prove that if D' is obtained by a PA-derivation D in sequent calculus by one step of Gentzen's reduction algorithm in his consistency proof of PA, as described in Takeuti (1987), then it is possible to obtain $H(D')$ from $H(D)$ by a finite number of steps of the Hydra Game.

Other examples of so-called "natural independence phenomena", which are considered by most logicians as more natural than the metamathematical incompleteness results first discovered by Gödel, are the powerful *tree theorem* due to Kruskal, as well as its *finite miniaturization* due to Harvey Friedman. These versions of Kruskal's theorem are remarkable from a proof-theoretic point of view because they are not provable in relatively strong logical systems (see e.g. Simpson (1990) and Gallier (1991)). Kruskal's theorem on trees is a classical result of combinatorics with several applications in computer science. The formal system we are interested in here is the second order system called "Arithmetical Transfinite Recursion" ATR_0 and is one of the "big five" systems well known in the area of *Reverse Mathematics* (see the classic treatise Simpson (1999)).