CLAUDIO BORRI
MICHELE BETTI, ENZO MARINO

Lectures on solid mechanics

Firenze University Press
2008

http://digital.casalini.it/9788884538543

ISBN 978-88-8453-854-3 (online)
ISBN 978-88-8453-853-6 (print)

This publication has been supported by the DEREC Tempus Project with the contribution of the European Commission Directorate General for Education and Culture and by Ente Cassa di Risparmio di Firenze.

Progetto grafico di Alberto Pizarro Fernández

© 2008 Firenze University Press
Università degli Studi di Firenze
Firenze University Press
Borgo Albizzi, 28
50122 Firenze, Italy
http://www.fupress.com/

Printed in Italy
# Contents

List of Figures ix  
Foreword XIII  

Part I  
Theory of elasticity 1  

---  

## Chapter 1 – Outline of linear algebra  
1.1 Vector spaces and linear mappings 3  
1.1.1 Vector spaces 3  
1.1.2 Linear mappings 4  
1.2 Euclidean spaces 8  
1.2.1 Euclidean metric tensor and scalar product 9  
1.2.2 Eigenvalues and eigenvectors 10  
1.3 Tensors 11  
1.3.1 Tensors and linear mappings 11  
1.4 Coordinate systems 13  
1.4.1 Linear mappings and the metric tensor 14  
1.4.2 Examples of coordinate systems 17  
1.4.3 Volumes and the vector product 21  
1.5 Covariant differentiation 24  
1.5.1 Grad, div, curl and Laplace’s operator 25  
1.6 Affine space 29  
1.6.1 Free and applied vectors 29  
1.7 Surfaces 32  

## Chapter 2 – Analysis of strain  
2.1 Introduction 37  
2.2 Deformation 38  
2.3 Strain tensor in general coordinates 38  
2.3.1 Examples of strain in Cartesian coordinates 44  
2.3.2 Infinitesimal deformations 48  
2.3.3 Deformation and rigid body motion 49  
2.4 Shell continuum 51  

2.4.1 General assumptions 51
2.4.2 Strain tensor 52

Chapter 3 – Analysis of stress 57
3.1 Body and surface forces 57
3.2 State of stress 58
3.2.1 Stress vector components 61
3.2.2 Stress tensor 61
3.3 Equations of equilibrium 64
3.3.1 Translational equilibrium 64
3.3.2 Rotational equilibrium 65
3.4 Principal stresses and principal directions 67
3.4.1 Normal and tangential components of the stress vector 69
3.4.2 Mohr’s circles 70
3.5 Stress quadric of Cauchy 77
3.6 Stress-deviator and spherical components of the stress tensor 78
3.7 Stress in shell continuums 79
3.7.1 Shifters 79
3.7.2 Contraction of surface forces 80
3.7.3 Body forces and load density 84
3.7.4 Euler’s equations 85
3.7.5 Membrane state of stress 88

Chapter 4 – Equations of elasticity 89
4.1 The material law 89
4.1.1 Generalized Hooke’s law 90
4.2 The linear elastic problem 95
4.2.1 Boundary value problem in terms of stresses 96
4.2.2 Boundary value problem in terms of displacements 98
4.3 Constitutive equation for shell continuums 100

Chapter 5 – Principle of Virtual Work 103
5.1 Virtual work 103
5.1.1 A simple example 107
5.2 PVW, Compatibility conditions, Equilibrium 109

Chapter 6 – Energy principles and variational methods 111
6.1 The strain-energy function and Hooke’s law 111
6.1.1 Superposition principle 116
6.1.2 Uniqueness of the solution 117
6.1.3 Theorem of reciprocity 119
6.2 Variational methods
   6.2.1 Potential energy 120
   6.2.2 Complementary energy 122
   6.2.3 Theorems of Castigliano 124

Chapter 7 – Strength of materials
   7.1 Introduction 129
   7.2 Maximum stress theory 130
   7.3 Maximum strain theory 131
   7.4 Beltrami’s theory 132
   7.5 Von Mises’ criterion 132
   7.6 Criteria comparison 134
      7.6.1 Maximum stress 134
      7.6.2 Maximum strain 135
      7.6.3 Beltrami’s criterion 136
      7.6.4 Von Mises’ criterion 137
      7.6.5 Comparison 138

Part II. Theory of elastic beams

Chapter 8 – Saint-Venant’s problem
   8.1 Statement of the problem 143
      8.1.1 External and internal forces 148
   8.2 Four fundamental cases 153
   8.3 Beam under axial force
      8.3.1 State of stress 154
      8.3.2 State of strain 155
      8.3.3 Displacement field 155
      8.3.4 Strain energy 157
   8.4 Beam under terminal couples
      8.4.1 Introductive sketch 158
      8.4.2 State of stress 159
      8.4.3 State of strain 162
      8.4.4 Displacement field 164
      8.4.5 Strain energy 167
   8.5 Beam under torsional couples
      8.5.1 Circular bar 169
      8.5.2 Cylindrical bar 172
      8.5.3 State of strain 176
      8.5.4 Displacement field 176
      8.5.5 Strain energy 178
      8.5.6 Torsion of tubular beams: Bredt’s theory 181
   8.6 Bending and shear 184
8.6.1 External forces 184
8.6.2 State of normal stress 185
8.6.3 State of tangential stress: Jourawski’s theory 187
8.6.4 Tangential stress for symmetrical cross-sections 189
8.6.5 State of strain 193
8.6.6 Total strain energy 197
8.6.7 Rectangular cross-section 198

Part III. Appendix  201

A – Applications of the shell theory 203
A.1 Spherical dome 203
  A.1.1 Geometry 203
  A.1.2 Displacements and strains 204
  A.1.3 Equilibrium and constitutive law 205
A.2 Cylindrical shell 208
  A.2.1 Geometry 208
  A.2.2 Displacements and strains 209
  A.2.3 Equilibrium and constitutive law 209
A.3 Hyperboloid of one sheet 211
  A.3.1 Geometry 211
  A.3.2 Equilibrium 215
List of figures

1.1 Contravariant and covariant bases related to a 2D
curvilinear coordinate system 14
1.2 Cylindrical coordinate system 18
1.3 Spherical coordinate system 20
1.4 Addition of two applied vectors 30
1.5 Subtraction of two applied vectors 31
1.6 Vector product for Cartesian applied vectors 32

2.1 Unstrained and strained body states 40
2.2 Measure of strain 42
2.3 Angular dilatation 45
2.4 Area dilatation 46
2.5 Two dimensional sketch of the displacement field for
Kirchho-Love shells 53

3.1 Body and surface forces 58
3.2 Body $V$ being in an equilibrium state 59
3.3 Splitting of the continuous media $V$ 60
3.4 Stress vectors: the sketch of Cauchy’s theorem 62
3.5 Stress tensor components 63
3.6 Stress tensor components acting on an in nitesimal
volume element 64
3.7 Plane $(y, z)$. Components of the stress tensor acting
on the volume element 66
3.8 Normal and tangential components of the stress
vector 69
3.9 Normal and tangential components of the stress vector
in two dimensions 71
3.10 Normal and tangential components of the stress vector
for in two dimensions 73
3.11 Graphical determination of principal directions 74
3.12 Mohr’s circles 76
3.13 Stress quadratic of Cauchy 77
3.14 Local bases in $G(\varepsilon)$ and in $Q$ 81
<table>
<thead>
<tr>
<th>FIGURE</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Hooke’s law</td>
</tr>
<tr>
<td>5.1</td>
<td>Forces and constraints acting on the continuous</td>
</tr>
<tr>
<td>5.2</td>
<td>Elemental volume element</td>
</tr>
<tr>
<td>5.3</td>
<td>Virtual deformation</td>
</tr>
<tr>
<td>5.4</td>
<td>Example</td>
</tr>
<tr>
<td>6.1</td>
<td>Density of strain energy in the case of axial state of stress</td>
</tr>
<tr>
<td>6.2</td>
<td>The simplest application of Clapeyron’s theorem</td>
</tr>
<tr>
<td>6.3</td>
<td>Forces and displacements acting on the body ( \nu ) lying in the equilibrium state</td>
</tr>
<tr>
<td>6.4</td>
<td>Concentrated loads acting on the body ( \nu )</td>
</tr>
<tr>
<td>6.5</td>
<td>Example of Castigliano’s theorems</td>
</tr>
<tr>
<td>7.1</td>
<td>Rupture domain for the maximum stress criterion</td>
</tr>
<tr>
<td>7.2</td>
<td>Rupture domain for the maximum strain criterion</td>
</tr>
<tr>
<td>7.3</td>
<td>Elastic domain for Beltrami’s criterion</td>
</tr>
<tr>
<td>7.4</td>
<td>Elastic domain for the Mises’ criterion</td>
</tr>
<tr>
<td>8.1</td>
<td>Prototype of beam</td>
</tr>
<tr>
<td>8.2</td>
<td>Unit normal vectors on the bases of the cylinder</td>
</tr>
<tr>
<td>8.3</td>
<td>Equilibrated components of force and couple resultants acting on the ends of the beam</td>
</tr>
<tr>
<td>8.4</td>
<td>Strained state of a beam subjected to an axial force</td>
</tr>
<tr>
<td>8.5</td>
<td>Strained state of a beam subjected to an axial force: radial contraction</td>
</tr>
<tr>
<td>8.6</td>
<td>Beam under terminal couples</td>
</tr>
<tr>
<td>8.7</td>
<td>Projection of the couples and rotation axis</td>
</tr>
<tr>
<td>8.8</td>
<td>Neutral axis and flexural axis</td>
</tr>
<tr>
<td>8.9</td>
<td>Circular bar under torsional couples</td>
</tr>
<tr>
<td>8.10</td>
<td>Rotation of a point ( p ) lying on a generic cross section of the circular beam</td>
</tr>
<tr>
<td>8.11</td>
<td>The sub-domain ( A_c ) bounded by the curve ( c )</td>
</tr>
<tr>
<td>8.12</td>
<td>Stress flux within a small region included by two closed curves and two generic transversal sections</td>
</tr>
<tr>
<td>8.13</td>
<td>Stress resultants</td>
</tr>
<tr>
<td>8.14</td>
<td>Shear regions</td>
</tr>
<tr>
<td>8.15</td>
<td>Beam splitting</td>
</tr>
<tr>
<td>8.16</td>
<td>Symmetrical cross section</td>
</tr>
<tr>
<td>8.17</td>
<td>Maximum shear stress for symmetrical cross section</td>
</tr>
<tr>
<td>8.18</td>
<td>( \sigma_3x ) distribution for symmetrical cross sections</td>
</tr>
<tr>
<td>8.19</td>
<td>Bending strain for an infinitesimal beam segment</td>
</tr>
<tr>
<td>8.20</td>
<td>Shear strain for an infinitesimal beam segment</td>
</tr>
</tbody>
</table>
8.21 Two contributions to the state of strain for a beam subjected to terminal forces 197

A.1 Hyperbolic coordinate system 212
Foreword

These Lecture Notes introduce the theoretical basics of solid mechanics to environmental engineering students. Born out of and supported by the European Project DEREC TEMPUS JEP Development of Environmental and Resources Engineering Curriculum, it collects the lectures held by the Authors during the course of Mechanic of Solids at the University of Florence, Degree of Environmental Engineering and Resources. Although the course is extended to basic structural engineering principles, such as mechanics, statics, kinematics and fundamental equations of beam structures, inertia, iso static and hyper static solution methods, these Lecture Notes reflect only the content of the lectures of continuum mechanics.

Several approaches are possible to the subject depending on the concern, either mathematically or physically oriented. The volume aims to provide a synthesis of both approaches, presenting in an organic whole the classical theory of solid mechanics and a more direct engineering approach. It is the Authors’ opinion that a top-down learning process may offer to the engineering students those critical and autonomy tools necessary to gain awareness of that continuous learning process that is required; it characterizes the cultural and technical personality of an engineer. An ongoing learning is all the more necessary today, where the rapid development of powerful computers and computer solving methods (finite element methods, discrete volume methods, boundary methods, etc.) have opened up the way to new horizons that the classical approaches were only able to formulate. This fast and impressive growth of computer methods seems to be replacing the importance of gaining a consolidated knowledge of solid mechanics background. On the contrary, the Authors believe that only a conscious knowledge of theory can be that cultural instrument through which an engineer can really hope to control the use of computer methods. With this aim, the Reader addressed by this volume is mainly the undergraduate student in Engineering Schools: it is organized in eight Chapters: Chapter 1 proposes a synthesis of the basic concepts of mathematics and geometry that the readers need in the following chapters. Chapter 2 and Chapter 3 are devoted to the elementary framework of strain and stress in an elastic body. The concept of finite strain and Cauchy stress state is introduced, together with Mohr’s representation of a general state of stress. Chapter 4 focuses on the classical law of linear elasticity. Chapter 5 deals with the Principle of Virtual Works. Chapter 6 treats the energy principles and provides a basic introduction to the variational methods.
Finally, Part I ends with a chapter introducing the notion of strength of materials. At the end of each chapter of the first part the basics of the tensor–based shell theory are also presented and then an application to some standard shell geometries is provided in appendix A.

The second part, Chapter 8, is dedicated to De Saint-Venant’s problem where the classical beam theory is presented focusing on the four fundamental cases: beam under axial forces, terminal couples, torsion, bending and shear.

The volume, that consolidates the Lecture Notes prepared by the Authors for the second–year undergraduate students in environmental engineering, proposes a widening of the classical theories approached, giving a list of references used during its preparation as a possible suggestion to the Reader.

The Authors wish to express their heartfelt gratitude to professor Marco Modugno for the inspiring discussions and stimulating suggestions.

It is also our pleasure to thank Eng. Seymour Milton John Spence for kindly revising the English text.

The publication of this book has been possible thanks to the financial support of the European Commission (DEREC Tempus Project) and Ente Cassa di Risparmio di Firenze to whom the Authors are extremely grateful.

Claudio Borri, Michele Betti, Enzo Marino
PART I

Theory of elasticity
Chapter 1
Outline of linear algebra

This chapter briefly presents some preliminary mathematics necessary to understand continuum mechanics. To this end the basic concepts of linear algebra and tensor analysis will be introduced. At the end of the chapter an overview of the theory of surfaces will be exposed in order to make the reader familiar with some background required for the mechanics of shell continuums, even though the latter is not the key theme of this book.

This introduction is neither exhaustive nor complete; indeed for any further insight the reader is warmly recommended to refer to the main sources from which this summary has been derived: Modugno, [4] and [5]; Sokolnikoff, [1]; Green-Zerna, [3].

1.1 Vector spaces and linear mappings

1.1.1 Vector spaces

We define vector space a set $\bar{V}$ equipped with the following operations

\[ + : \bar{V} \times \bar{V} : (\bar{u}, \bar{v}) \mapsto \bar{u} + \bar{v} \quad (1.1) \]
\[ \cdot : \mathbb{R} \times \bar{V} : (\lambda, \bar{v}) \mapsto \lambda \bar{v}. \quad (1.2) \]

Elements belonging to $\bar{V}$ are named vectors and are characterized by the following properties

1. $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$ \hspace{1cm} $\forall \bar{u}, \bar{v}, \bar{w} \in \bar{V}$
2. $\bar{u} + \bar{v} = \bar{v} + \bar{u}$ \hspace{1cm} $\forall \bar{u}, \bar{v} \in \bar{V}$
3. $\bar{u} + \bar{0} = \bar{u}$ \hspace{1cm} $\forall \bar{u} \in \bar{V}$
4. $\forall \bar{u} \in \bar{V}$ $\exists \bar{v}$ \hspace{1cm} $\bar{u} + (-\bar{u}) = \bar{0}$ so that $\bar{u} + (-\bar{u}) = \bar{0}$
where $\vec{0}$ is called null vector.

Every vector space admits the existence of a subset

$$\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\} \subset \vec{V}$$

called the basis of $\vec{V}$. Thus, each vector $\vec{v} \in \vec{V}$ can be univocally represented through the basis $\mathcal{B}$ as follows

$$\vec{v} = v^i \vec{b}_i \quad i = 1, \ldots, n$$

(1.3)

where $v^i \in \mathbb{R}$ are the components of $\vec{v}$ related to the basis $\mathcal{B}$ and $n$ is a number which defines the dimension of $\vec{V}$, namely the number of vectors in any basis of $\vec{V}$.

Notice that in equation (1.3) the Einstein’s summation convention has been used. It is a notational convenience where any term in which an index appears twice will stand for the sum of all such terms as the index assumes all of a preassigned range of values, hence

$$\vec{v} = v^i \vec{b}_i = \sum_{i=1}^{n} v^i \vec{b}_i$$

(1.4)

### 1.1.2 Linear mappings

Functions between two vector spaces assume a crucial importance in linear algebra. In particular, we define a linear map as a linear transformation between two vector spaces that preserves the operations of vector addition and scalar multiplication.

Let $\vec{V}$ and $\vec{V}'$ be two vector spaces equipped with the bases

$$\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}, \quad \mathcal{B}' = \{\vec{b}'_1, \ldots, \vec{b}'_m\}$$

respectively.

We define a linear mapping as the transformation

$$f : \vec{V} \rightarrow \vec{V}', \quad \vec{v} \mapsto \vec{v}'$$

(1.5)

if the two following conditions are satisfied

1. $f (\vec{u} + \vec{v}) = f (\vec{u}) + f (\vec{v}) \quad \forall \vec{u}, \vec{v} \in \vec{V}$: additivity;
2. $f (\lambda \vec{u}) = \lambda f (\vec{u}) \quad \forall \vec{u} \in \vec{V} \in \mathbb{R}$: homogeneity.
The set of all linear maps from \( \vec{V} \) to \( \vec{V}' \), denoted by \( L (\vec{V}, \vec{V}') \), represents a \( n \times m \)-dimensional vector space, where \( n \) and \( m \) are the dimensions of \( \vec{V} \) and \( \vec{V}' \), respectively.

\[
\{ f : \vec{V} \to \vec{V}' \} =: L (\vec{V}, \vec{V}') \tag{1.6}
\]

For linear mappings the following properties hold

1. \((f + g)(\vec{u}) = f(\vec{u}) + g(\vec{u}), \quad \forall f, g \in L (\vec{V}, \vec{V}'); \quad \vec{u} \in \vec{V} \)
2. \((\lambda f)(\vec{u}) = \lambda f(\vec{u}), \quad \forall f \in L (\vec{V}, \vec{V}'); \quad \vec{u} \in \vec{V} \)

**Matrix representation**

Notions so far introduced allow us to assert that if \( f \) is a linear mapping from \( \vec{V} \) to \( \vec{V}' \), then \( f(\vec{v}) \) is a vector in \( \vec{V}' \). Consequently, by recalling the expression in components for \( \vec{v} \), (1.3), we have

\[
f(\vec{v}) = f(\vec{v})^i \vec{b}_i^i \quad i = 1, \ldots, m \tag{1.7}
\]

and accounting for the fact that \( \vec{v} = v^j \vec{b}_j \), with \( j = 1, \ldots, n \), and by using the homogeneity property for linear mappings, the latter equation leads to

\[
f(v^j \vec{b}_j)^i \vec{b}_i^i = v^j f(\vec{b}_j)^i \vec{b}_i^i \quad j = 1, \ldots, n \quad i = 1, \ldots, m. \tag{1.8}
\]

In a shorter form the components of \( f(\vec{v}) \) are then

\[
(f(\vec{v}))^i = f_j^i v^j \tag{1.9}
\]

so that the \( m \times n \)-dimensional matrix \( f_j^i = f(\vec{b}_j)^i \) is the matrix representation of the linear mapping \( f \) referred to the bases \( \mathcal{B} \) and \( \mathcal{B}' \).

**Linear forms and the dual space**

Linear forms are a special case of linear mappings. Let \( \vec{V} \) be a vector space and \( \mathcal{B} = \{ \vec{b}_i \} \) its basis. A linear form \( \omega \) is a linear transformation from \( \vec{V} \) to a scalar field

\[
\omega : \vec{V} \to \mathbb{R} \tag{1.10}
\]

Hence, we define \( \vec{V}^* \) as the set of linear forms from \( \vec{V} \) to \( \mathbb{R} \)

\[
\vec{V}^* =: \{ \omega : \vec{V} \to \mathbb{R} \} =: L (\vec{V}, \mathbb{R}) \tag{1.11}
\]
$V^*$ and $\bar{V}$ have the same dimension.

The dual space $V^*$ admits a basis $\mathcal{B}^* = \{\beta^i\}$ whose elements are linear forms operating as follows

$$\beta^i (\bar{b}_j) = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(1.12)

By the definition, we can state that the element $\beta^i$ belonging to $\mathcal{B}^*$, applied to the vector $\bar{u}$, yields a scalar that is the $i$-th component of $\bar{u}$. In fact we write

$$\beta^i (\bar{u}) = \beta^i (u^j \bar{b}_j) = u^j \delta^i_j = u^i$$

(1.13)

We highlight that, as done for vectors, each linear form, chosen the $n$-dimensional basis $\mathcal{B}^*$, can be written in components as follows

$$\omega = \omega_j \beta^j \quad j = 1, \ldots, n$$

(1.14)

**Bilinear forms**

We can define a bilinear form $f$ as a mapping

$$f : \bar{V} \times \bar{V} \to \mathbb{R}, \quad (\bar{v}, \bar{v}') \mapsto \lambda$$

(1.15)

where $\bar{v}, \bar{v}' \in \bar{V}$ and $\lambda \in \mathbb{R}$, and such that it is linear in each argument separately. That is

1. $f (\bar{v} + \bar{w}, \bar{v}') = f (\bar{v}, \bar{v}') + f (\bar{w}, \bar{v}');$
2. $f (\bar{v}, \bar{v}' + \bar{w}) = f (\bar{v}, \bar{v}') + f (\bar{v}', \bar{w});$
3. $f (\lambda \bar{v}, \bar{v}) = f (\bar{v}, \lambda \bar{v}) = \lambda f (\bar{v}, \bar{v}').$

$\forall f \in L (\bar{V} \times \bar{V}, \mathbb{R}) ; \bar{v}, \bar{v}', \bar{w} \in \bar{V}; \lambda \in \mathbb{R}.$

**Endomorphisms**

Frequently in the field of solid mechanics we will meet special linear mappings from a vector space into itself, i.e. $f \in L (\bar{V}, \bar{V})$. These are defined endomorphisms

$$f : \bar{V} \to \bar{V}, \quad \bar{v} \mapsto \bar{v}' \quad \bar{v}, \bar{v}' \in \bar{V}$$

(1.16)

The set of linear mappings from $\bar{V}$ into itself forms a $n \times n$-dimensional vector space, where $n$ is the dimension of $\bar{V}$.

$$\{f : \bar{V} \to \bar{V}\} =: L (\bar{V}, \bar{V}) =: \text{End} (\bar{V})$$

(1.17)
Change of basis for endomorphisms

Let $\mathcal{B}$ be a fixed basis for $\bar{V}$, we are interested in evaluating how the endomorphism $f \in \text{End}(\bar{V})$ changes when passing to a new basis $\mathcal{B}'$ of $\bar{V}$. The following transformation rules are established

\begin{align*}
\bar{b}_i &= a_{ih}^l \bar{b}_h' \\
\bar{b}_h' &= a_{jh}^l \bar{b}_j
\end{align*}

(1.18) (1.19)

that, by replacing (1.19) into (1.18), yield

\begin{align*}
\bar{b}_i &= a_{ih}^l a_{hk}^k \bar{b}_k
\end{align*}

(1.20)

and so

\begin{align*}
\left( a_{ih}^l a_{hk}^k - \delta_i^k \right) \bar{b}_k = 0 \Rightarrow a_{ih}^l a_{hk}^k &= \delta_i^k
\end{align*}

(1.21)

therefore, each change of basis for $\bar{V}$ is characterized by a square invertible matrix $n \times n$.

Likewise vectors, the following rules hold for dual elements

\begin{align*}
\bar{\beta}^i &= a_{hi}^l \bar{\beta}'^h \\
\bar{\beta}'^i &= a_{hi}^l \bar{\beta}^h
\end{align*}

(1.22) (1.23)

When both bases are orthogonal, then the transformation matrices are also orthogonal, that is

\begin{align*}
a_{ih}^l &= a_{hi}^l
\end{align*}

(1.24)

where $a_{ih}^l = \left(a_{hi}^l\right)^{-1}$, and

\begin{align*}
a_{ji}^l &= \cos(\bar{b}_i', \bar{b}_j) \\
a_{hi}^h &= \cos(\bar{b}_h, \bar{b}_h')
\end{align*}

(1.25) (1.26)

The change of basis implies a change of the vector components. In fact we have

\begin{align*}
v^k &= a_{jk}^h v^j \\
v'^k &= a_{jk}^h v^j
\end{align*}

(1.27) (1.28)
The proof of the above equations can be easily provided. For instance, for equation (1.27) we have that a vector \( \bar{v} \) can be expressed with respect to two basis \( \mathcal{B} \) and \( \mathcal{B}' \) as \( \bar{v} = v^i \bar{b}_i = v'^j \bar{b}'_j \). Hence

\[
v^i \bar{b}_i = v'^j a^k_j \bar{b}_k \Rightarrow v'^j a^k_j \bar{b}_k - v^i \bar{b}_i = 0 \Rightarrow \quad (1.29)
\]

\[
v'^j a^k_j \bar{b}_k - v^i \delta^k_i \bar{b}_k = 0 \Rightarrow v'^j a^k_j - v^i \delta^k_i \bar{b}_k = 0 \Rightarrow \quad (1.30)
\]

finally, by putting zero the coefficient in brackets, we obtain relation (1.27).

Covector components change by the following rules

\[
v_k = a^i_k v'_i \quad (1.31)
\]

\[
v'_k = a^i_k v_i \quad (1.32)
\]

Furthermore, recalling equation (1.9), via some manipulations, we get the rule to transform the endomorphism \( f \), that is

\[
f'^i_j = a^i_h f^h_k a^l_j \quad \Rightarrow \quad (1.33)
\]

and

\[
f'^h_j = a^i_h f^h_k a^j_k \quad (1.34)
\]

Similar relationships can be found for higher order matrices, for instance for a mixed fourth-order tensor we have

\[
f'^{ij}_{hk} = a^i_l a^j_m f^{lm}_{no} a^m_h a^l_o \quad \Rightarrow \quad (1.35)
\]

and likewise

\[
f'^{ij}_{hk} = a^i_l a^j_m f^{lm}_{no} a^h_n a^l_k \quad (1.36)
\]

### 1.2 Euclidean spaces

A Euclidean vector space is a space which admits a Euclidean metric, that is a structure inducing some special relationships between distances and angles. In particular, fixed a Cartesian coordinate system (that will be better defined later on) and its standard basis, in a Euclidean space the distance between two points can be computed by means of Pitagora’s formula.

\footnote{Often, within an engineering context, it is convenient to represent equations (1.33) and (1.34) in the matrix form, such as \( F' = R^T FR \) and \( F = R F' R^T \), where \( R^T \) and \( R \) are nothing but \( a^i_j \) and \( a^k_h \), respectively.}
1.2.1 Euclidean metric tensor and scalar product

Let $\bar{V}$ be an $n$-dimensional vector space and $B = \{\bar{b}_i\}$ be its basis. We define Euclidean metric the symmetric positive definite bilinear mapping

$$g : \bar{V} \times \bar{V} \rightarrow \mathbb{R}$$

(1.37)

that, given a pair of vectors $\bar{u}, \bar{v} \in \bar{V}$, gives a real number $g(\bar{u}, \bar{v})$ as follows

$$\bar{u} \cdot \bar{v} =: g(\bar{u}, \bar{v})$$

(1.38)

The number $g(\bar{u}, \bar{v})$ is termed scalar product. The Euclidean metric allows us to compute distances. Indeed, we define length (or modulus, or norm) of $\bar{v} \in \bar{V}$ the real number

$$||\bar{v}|| = \sqrt{g(\bar{v}, \bar{v})} \geq 0$$

(1.39)

The angle $\vartheta$ amid vectors $\bar{u}$ and $\bar{v}$ is given by the following equation

$$\cos \vartheta = \frac{g_{ij} u^i v^j}{\sqrt{|g_{ij} u^i v^j| |g_{ij} u^i v^j|}}$$

(1.40)

To compute the components of the metric tensor, i.e. the matrix representing the mapping $g$, given the basis $B = \{\bar{b}_i\}$ of $\bar{V}$, the following general rule is adopted

$$g_{ij} = g(\bar{b}_i, \bar{b}_j) = \bar{b}_i \cdot \bar{b}_j$$

(1.41)

that in the expanded form becomes

$$g_{ij} = \begin{pmatrix}
\bar{b}_1 \cdot \bar{b}_1 & \cdots & \bar{b}_1 \cdot \bar{b}_n \\
\vdots & \ddots & \vdots \\
\bar{b}_n \cdot \bar{b}_1 & \cdots & \bar{b}_n \cdot \bar{b}_n
\end{pmatrix}$$

(1.42)

In the light of the above general expression for the metric tensor, the scalar product between two vectors becomes

$$\bar{u} \cdot \bar{v} = u^i \bar{b}_i \cdot v^j \bar{b}_j = u^i w^j \bar{b}_i \cdot \bar{b}_j = g_{ij} u^i v^j$$

(1.43)

Expression (1.43) includes, of course, the special case when, fixed a Cartesian coordinate system, the metric matrix equals the identity matrix $\delta_{ij}$ and consequently the scalar product can be carried out multiplying component by component, i.e. $\bar{u} \cdot \bar{v} = u^1 v^1 + \cdots + u^n w^n$. 


Now we want to point out that between the \(n\)-dimensional vector space \(\mathcal{V}\) and its dual \(\mathcal{V}^*\) there exists an isomorphism. Note that we are using some special words, e.g. isomorphism, without giving the formal mathematical definition. This lies beyond the purpose of this book, so that, also in this case, we will restrict the current exposition to an intuitive concept. From this point of view, an isomorphism is a one-to-one mapping of an algebraic structure, e.g. vector space, into another of the same type, preserving all algebraic relations.

Thus we define the musical isomorphisms: flat and sharp, respectively, as follows

\[
\begin{align*}
g^\flat &: \mathcal{V} \to \mathcal{V}^*: \overline{v} \mapsto v \\
g^\sharp &: \mathcal{V}^* \to \mathcal{V}: v \mapsto \overline{v}
\end{align*}
\]  

(1.44) (1.45)

where

\[
\delta (v) = g (\overline{u}, \overline{v}) , \quad \forall \overline{u} \in \mathcal{V}
\]  

(1.46)

The isomorphism between \(\mathcal{V}\) and \(\mathcal{V}^*\) implies the existence of a metric tensor

\[
\bar{g}: \mathcal{V}^* \times \mathcal{V}^* \to \mathbb{R}
\]  

(1.47)

so that

\[
\bar{u} \cdot \bar{v} = \bar{g} (\overline{u}, \overline{v}) := \bar{g} (u, v) = u \cdot v
\]  

(1.48)

For further details the reader is referred to [4].

Both \(g^\flat\) and \(g^\sharp\) are particularly helpful when carrying out computations it is necessary to switch from the contravariant form to the covariant form (and vice versa); namely when we need to lower or raise the indices.

### 1.2.2 Eigenvalues and eigenvectors

Let \(\mathcal{V}\) be a \(n\) dimensional vector space and \(\mathcal{B} = \{ \overline{b}_1, \ldots, \overline{b}_n \}\) the vector basis. Given \(f \in \text{End} (\mathcal{V})\), we define the eigenvector a nonzero vector \(\overline{v}\) whose direction does not change under the effect of \(f\). Formally

\[
f (\overline{v}) = \lambda \overline{v}, \quad \lambda \in \mathbb{R}
\]  

(1.49)

When equation (1.49) holds we can also define the real number \(\lambda\) as the eigenvalue for \(\overline{v}\).
The eigenvalues of \( f \) represent the real roots of the following polynomial

\[
p_n(\lambda) = \det (f^i_j - \lambda \delta^i_j)
\]  

where \( p_n(\lambda) \) is called the characteristic polynomial of degree \( n \) and \( f^i_j \) is the matrix representation of the endomorphism \( f \).

### 1.3 Tensors

This section is devote to a short outline of tensor analysis.

Given two vector spaces \( \bar{U} \) and \( \bar{V} \) it is possible to construct a new structure, i.e. a third vector space, called tensor product of \( \bar{U} \) times \( \bar{V} \) that is symbolically denoted by \( \bar{U} \otimes \bar{V} \). This vector space is made up of elements called tensors. It is possible to demonstrate that if

\[
\mathcal{B}_{\bar{U}} = \{ \bar{u}_1, \ldots, \bar{u}_n \}
\]

\[
\mathcal{B}_{\bar{V}} = \{ \bar{v}_1, \ldots, \bar{v}_m \}
\]

are bases for \( \bar{U} \) and \( \bar{V} \), respectively, then

\[
\mathcal{B}_{\bar{U} \otimes \bar{V}} = \{ \bar{u}_i \otimes \bar{v}_j \}, \ i = 1, \ldots, n; \ j = 1, \ldots, m
\]

is a basis of the vector space \( \bar{U} \otimes \bar{V} \). Therefore, each tensor \( \bar{\tau} \in \bar{U} \otimes \bar{V} \) can be univocally expressed by

\[
\bar{\tau} = \bar{\tau}^{ij} (\bar{u}_i \otimes \bar{v}_j)
\]  

(1.51)

where again the Einstein’s summation convention has been used, in fact (1.51) can also be written

\[
\bar{\tau} = \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{\tau}^{ij} \bar{u}_i \otimes \bar{v}_j
\]

### 1.3.1 Tensors and linear mappings

The definition of tensors does not alter the structure of \( \bar{U} \) and \( \bar{V} \), and, since the dual space \( \bar{V}^* \) preserves the structure of a vector space, we can introduce tensors belonging to spaces such as \( \bar{U}^* \otimes \bar{V}^* \)
and $\bar{U}^* \otimes \bar{V}$. In other words we distinguish the following second order tensors:

\[
\begin{align*}
\bar{U}^* \otimes \bar{V} : & \quad \text{mixed tensors} \\
\bar{U}^* \otimes \bar{V}^* : & \quad \text{covariant tensors} \\
\bar{U} \otimes \bar{V} : & \quad \text{contravariant tensors}
\end{align*}
\]

**Mixed tensors:** given the $n$–dimensional vector spaces $\bar{V}$ and $\bar{V}^*$, let $\alpha \in \bar{V}^*$ be a dual form and $\bar{v} \in \bar{V}$ a vector, then the tensor $\alpha \otimes \bar{v} \in \bar{V}^* \otimes \bar{V}$ can be identified by the endomorphism $\alpha \otimes \bar{v} \in \text{End} (\bar{V}) = L (\bar{V}, \bar{V})$ defined as

\[
\alpha \otimes \bar{v} : \bar{V} \rightarrow \bar{V} : \bar{u} \mapsto (\alpha \otimes \bar{v}) \bar{u} = \alpha (\bar{u}) \bar{v} \in \bar{V} \quad (1.52)
\]

Hence, a natural isomorphism has been obtained

\[
\bar{V}^* \otimes \bar{V} \cong L (\bar{V}, \bar{V}) \quad (1.53)
\]

**Covariant tensors:** Let $\alpha, \beta$ be two linear forms belonging to $\bar{V}^*$. We can identify the tensor $\alpha \otimes \beta \in \bar{V}^* \otimes \bar{V}^*$ by the bilinear form $\alpha \otimes \beta \in L^2 (\bar{V}, \mathbb{R})$ defined as

\[
\alpha \otimes \beta : \bar{V} \times \bar{V} \rightarrow \mathbb{R} : (\bar{u}, \bar{v}) \mapsto \alpha (\bar{u}) \beta (\bar{v}) \in \mathbb{R} \quad (1.54)
\]

Therefore we can realize another isomorphism, which is

\[
\bar{V}^* \otimes \bar{V}^* \cong L^2 (\bar{V}, \mathbb{R}) \quad (1.55)
\]

Vectors, linear forms and tensors so far discussed can be summarized in the following scheme

**Vectors**

\[
\bar{v} \in \bar{V} \quad (1.56)
\]

**Linear forms**

\[
\alpha \in \bar{V}^* \cong L (\bar{V}, \mathbb{R}) \quad (1.57)
\]

**II–order mixed tensors**

\[
\alpha \otimes \bar{v} \in \bar{V}^* \otimes \bar{V} \cong \text{End} (\bar{V}) \quad (1.58)
\]

**II–order covariant tensors**

\[
\alpha \otimes \beta \in \bar{V}^* \otimes \bar{V}^* \cong L^2 (\bar{V}, \mathbb{R}) \quad (1.59)
\]
1.4 Coordinate systems

Within the three-dimensional affine Euclidean space $E$ it is possible to define a coordinate system through the following bijections

$$X : E \rightarrow \mathbb{R}^3 \quad X^{-1} : \mathbb{R}^3 \rightarrow E$$

(1.60)

where $X = (x^1, x^2, x^3)$. The injectivity of $X$ assures the one-to-one correspondence between points belonging to $E$ and their coordinates. Namely, given a point $p \in E$ there exists the triplet $(x^1, x^2, x^3)$ which identifies such a point. The mapping $X$ is assumed to be differentiable as many times as required.

The coordinate system $X$ is made up of coordinate functions

$$x^i : E \rightarrow \mathbb{R}^i = 1, 2, 3$$

(1.61)

Moreover, we define the coordinate curves as the following mappings

$$x_{ip} : \mathbb{R} \rightarrow E \quad i = 1, 2, 3$$

(1.62)

such as

$$x_{1p}(\lambda) = X^{-1}(x^1(p) + \lambda, x^2(p), x^3(p))$$
$$x_{2p}(\lambda) = X^{-1}(x^1(p), x^2(p) + \lambda, x^3(p))$$
$$x_{3p}(\lambda) = X^{-1}(x^1(p), x^2(p), x^3(p) + \lambda)$$

that in a shorter form become

$$x^j(x_{ip}(\lambda)) = x^j(p) + \delta^j_i \lambda \quad p \in E, \quad \lambda \in \mathbb{R}$$

(1.63)

Given a point $p \in E$, there are three coordinate curves passing through it.

It is possible to demonstrate that the derivatives of the coordinate curves, computed for a fixed $\lambda$, are vectors forming a basis $B = \{\partial_i\}$ in $p$.

Analogously, it can be proved that the derivatives of the coordinate functions $\{x^i\}$ computed in $p$ form a covariant basis $B^* = \{d^i\}$ in such a point.

The above two bases satisfy the following relation

$$d^j(\partial_i) = \delta^j_i$$

(1.64)
Figure 1.1: Contravariant and covariant bases related to a 2D curvilinear coordinate system.

See [4] for further details.

Bases $B = \{\bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3\}$ and $B^* = \{d^1, d^2, d^3\}$ related to $X$ allow the representation of vectors, linear forms and tensor fields. For example we write

$$v = v^i \bar{\partial}_i, \quad \forall \bar{v} : E \rightarrow \bar{E} \quad (1.65)$$

$$w = w_i d^i \quad \forall w : E \rightarrow \bar{E}^* \quad (1.66)$$

where $\bar{v}$ and $w$ are vector and covector fields, respectively.

1.4.1 Linear mappings and the metric tensor

In order to represent an endomorphism $f$ by means of the coordinate system $X$ we can write

$$f = f^j_i d^j \otimes \bar{\partial}_j, \quad \forall f : E \rightarrow L(\bar{E}, \bar{E}) \cong \bar{E}^* \otimes \bar{E} \quad (1.67)$$

where

$$f^j_i = d^j (f(\bar{\partial}_i)) : E \rightarrow \mathbb{R} \quad (1.68)$$

and likewise, for the bilinear form we write

$$f = f_{ij} d^i \otimes d^j, \quad \forall f : E \rightarrow L^2(\bar{E}, \mathbb{R}) \cong \bar{E}^* \otimes \bar{E}^* \quad (1.69)$$

where

$$f_{ij} = f(\bar{\partial}_i, \bar{\partial}_j) : E \rightarrow \mathbb{R} \quad (1.70)$$

It is straightforward now to realize that the metric tensor $g$ is nothing but the following bilinear form

$$g : E \rightarrow L^2(\bar{E}, \mathbb{R}) \cong \bar{E}^* \otimes \bar{E}^* \quad (1.71)$$
indeed

\[ g = g_{ij} d^i \otimes d^j \quad (1.72) \]

where

\[ g_{ij} = g(\bar{\partial}_i, \bar{\partial}_j) \quad (1.73) \]

As a concluding remark of this section we point out the fact that once the coordinate system is fixed it is possible to find its vector basis, i.e. the covariant basis, and therefore the covariant expression of the metric tensor can be directly computed.

### 1.4.2 Components of the metric tensor

Suppose that \( X_e = \{x^i_c\}, \ i = 1, 2, 3 \) is a Cartesian coordinate system, with the origin \( o \in E \), which describes the affine Euclidean space \( E \) and \( \{\bar{e}_i\}, \ i = 1, 2, 3 \) its unit normal basis. Moreover, let \( X = \{x^j\}, \ j = 1, 2, 3 \) be a generic curvilinear coordinate and \( \{\bar{\partial}_j\}, \ j = 1, 2, 3 \) its basis. Suppose that the functions \( x^i_c \) and \( x^i \) are single-valued and continuously differentiable with respect to each of their variables as many times as required, we can therefore write

\[ x^i_c = x^i_c(x^1, x^2, x^3) \quad i = 1, 2, 3 \quad (1.74) \]

\[ x^i = x^i(x^1_c, x^2_c, x^3_c) \quad i = 1, 2, 3 \quad (1.75) \]

and the rules for changing basis (1.18) and (1.19) on page 7, become

\[ \bar{\partial}_i = \frac{\partial x^h_c}{\partial x^i} \bar{e}_h; \quad d^i = \frac{\partial x^i_c}{\partial x^h_c} e^h \quad (1.76) \]

and

\[ \bar{e}_i = \frac{\partial x^h_c}{\partial x^i_c} \bar{\partial}_h; \quad e^i = \frac{\partial x^i_c}{\partial x^h_c} d^h \quad (1.77) \]

where equations (1.76) transform the covariant and contravariant elements of the Cartesian basis into the elements of the curvilinear basis while expressions (1.77) perform the vice-versa.

Now, according to equation (1.41), it is possible to compute the covariant components of the metric tensor related to the curvilinear coordinate system

\[ g_{ij} = \bar{\partial}_i \cdot \bar{\partial}_j = \frac{\partial x^h_c}{\partial x^i} \bar{e}_h \cdot \frac{\partial x^k_c}{\partial x^j} \bar{e}_k = \]

\[ = \frac{\partial x^h_c}{\partial x^i} \frac{\partial x^k_c}{\partial x^j} \delta_{hk} = \frac{\partial x^h_c}{\partial x^i} \frac{\partial x^h_c}{\partial x^j} \quad (1.79) \]
Moreover, the contravariant components are

\[ g^{ij} = g^i \cdot g^j = \frac{\partial x^i}{\partial x^h_c} \bar{e}^h \cdot \frac{\partial x^j}{\partial x^k_c} \bar{e}^k = \]

\[ = \frac{\partial x^i}{\partial x^h_c} \frac{\partial x^j}{\partial x^k_c} \delta^{hk} = \frac{\partial x^i}{\partial x^h_c} \frac{\partial x^j}{\partial x^h_c} \]  

(1.80)

(1.81)

and finally the mixed components of the metric tensor are

\[ g^i_j = g^i (\bar{\partial}_j) = \frac{\partial x^i}{\partial x^h_c} \bar{e}^k = \]

\[ = \frac{\partial x^i}{\partial x^h_c} \frac{\partial x^k}{\partial x^j} \delta^h_k = \frac{\partial x^i}{\partial x^h_c} \frac{\partial x^k}{\partial x^h_c} \]  

(1.82)

(1.83)

**Christoffel symbols**

The Christoffel\(^2\)'s symbol are defined as follows

\[ \Gamma^k_{ij} = \frac{\partial}{\partial x^i} (\bar{\partial}_j) = E \rightarrow \mathbb{R} \]  

(1.84)

so that \( \nabla_i \bar{\partial}_j = \Gamma^k_{ij} \bar{\partial}_k \). Hence, \( \Gamma^k_{ij} \) is the \( k \)-th component of the derivative of the basis element \( \bar{\partial}_j \) along the \( i \)-th direction. Analytically they can be computed by the following formula

\[ \Gamma^k_{ij} = \frac{1}{2} g^{kh} (\partial_i g_{hj} + \partial_j g_{hi} - \partial_h g_{ij}) \]  

(1.85)

where \( \partial_i \) denotes the partial derivatives.

Moreover, it can be proved that

\[ \Gamma^k_{ij} = (\nabla_i \bar{\partial}_j)^k = - (\nabla_i d^k) \]  

(1.86)

For proofs and more details the reader is referred to [4], [5] and [1].

\( ^2\)Elwin Bruno Christoffel (November 10, 1829 Montjoie, now called Monschau - March 15, 1900 Strasbourg) was a German mathematician and physicist.

1.4.3 Examples of coordinate systems

**Cartesian coordinates**

The Cartesian coordinate system introduces considerable simplifications with respect to other curvilinear systems, e.g. cylindrical, spherical, hyperbolic, etc.

Therefore, let us begin by defining a *Cartesian coordinate system* as the triplet of coordinate functions

\[ X_c = (x, y, z) \equiv (x^1, x^2, x^3) : E \rightarrow \mathbb{R}^3 \]  

(1.87)

with an origin in \( o \in E \) and equipped with the unit normal basis, also called standard basis, \( \{\bar{e}_1, \bar{e}_2, \bar{e}_3\} \). Given \( p \in E \), the coordinate functions are such that

\[ x^i(p) =: (p - o) \cdot \bar{e}_i \]  

(1.88)

The coordinate curves of a Cartesian system are

\[ x_{ip}(\lambda) = p + \lambda \bar{e}_i \]  

(1.89)

Notice that for rectangular coordinate systems the symbols denoting the bases will turn into

\[ \bar{\partial}_i = \bar{e}_i \]  

(1.90)

\[ d^i = \bar{e}_i \]  

(1.91)

The covariant form of the metric tensor can be readily computed as follows

\[ g_{ij} = \bar{e}_i \cdot \bar{e}_j = \delta_{ij} \]  

(1.92)

Elements of the standard basis related to the Cartesian coordinate system do not vary with the point \( p \in E \). As a consequence, the Christoffel symbols are identically null.

\[ \Gamma^k_{ij} = 0 \]  

(1.93)

In addition to that we also highlight that here the upper or lower position of the indices does not influence the structure of the field we are dealing with. Namely, vectors and linear forms are the same and the unit normal basis equals its dual.

\[ g^b(\bar{e}_i) = \bar{e}^i = \bar{e}_i \]  

(1.94)
For this reason whenever given two sets of numbers having the same dimension, they can be ordered in a row and a column, respectively, and by using the multiplication rule row–by–column a scalar is always yielded without taking any care whether we are dealing with vectors or linear forms.

Cylindrical coordinates

We define a cylindrical coordinate system the functions

\[ X = (\rho, \vartheta, z) : E \rightarrow \mathbb{R}^3 \quad (1.95) \]

In this case, with the help of figure 1.2, equation (1.74) becomes

\[
\begin{align*}
x &= \rho \sin \vartheta \\
y &= \rho \cos \vartheta \\
z &= z
\end{align*}
\]

![Figure 1.2: Cylindrical coordinate system.](image)

Now, through equation (1.76), it is easy to compute the basis related to the cylindrical system

\[
\begin{align*}
\vec{\partial}_\rho &= \frac{\partial x}{\partial \rho} \vec{e}_1 + \frac{\partial y}{\partial \rho} \vec{e}_2 + \frac{\partial z}{\partial \rho} \vec{e}_3 \\
\vec{\partial}_\vartheta &= \frac{\partial x}{\partial \vartheta} \vec{e}_1 + \frac{\partial y}{\partial \vartheta} \vec{e}_2 + \frac{\partial z}{\partial \vartheta} \vec{e}_3 \\
\vec{\partial}_z &= \frac{\partial x}{\partial z} \vec{e}_1 + \frac{\partial y}{\partial z} \vec{e}_2 + \frac{\partial z}{\partial z} \vec{e}_3
\end{align*}
\]
Hence, the covariant components of the metric tensor is

\[ g_{\rho\rho} = 1 \]
\[ g_{\vartheta\vartheta} = \rho^2 \]
\[ g_{zz} = 1 \]
\[ g_{\vartheta z} = g_{\rho z} = g_{\rho\vartheta} = 0 \]

that in the matrix form can be written as follows

\[
g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.96)
\]

The contravariant form of the metric tensor is

\[
g^{ij} = (g_{ij})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.97)
\]

Using equation (1.85), the Christoffel symbols are

\[
\Gamma^\rho_{\vartheta\vartheta} = -\rho, \quad \Gamma^\vartheta_{\rho\rho} = \Gamma^\vartheta_{\rho\vartheta} = \frac{1}{\rho} \quad (1.98)
\]

**Spherical coordinates**

We define a spherical coordinate system by the functions

\[
X = (r, \vartheta, \varphi) : E \to \mathbb{R}^3 \quad (1.99)
\]

In this case, with the help of figure 1.3, equation (1.74) becomes

\[
x = r \sin \varphi \cos \vartheta \\
y = r \sin \varphi \sin \vartheta \\
z = r \cos \varphi
\]

Now, through equation (1.76), it is easy to compute the basis related to the spherical system

\[
\bar{\partial}_r = \frac{\partial x}{\partial r} \bar{e}_1 + \frac{\partial y}{\partial r} \bar{e}_2 + \frac{\partial z}{\partial r} \bar{e}_3 \\
\bar{\partial}_\vartheta = \frac{\partial x}{\partial \vartheta} \bar{e}_1 + \frac{\partial y}{\partial \vartheta} \bar{e}_2 + \frac{\partial z}{\partial \vartheta} \bar{e}_3 \\
\bar{\partial}_\varphi = \frac{\partial x}{\partial \varphi} \bar{e}_1 + \frac{\partial y}{\partial \varphi} \bar{e}_2 + \frac{\partial z}{\partial \varphi} \bar{e}_3
\]
Hence, the covariant components are

\[ g_{rr} = 1 \]
\[ g_{\vartheta\vartheta} = r^2 \sin^2 \varphi \]
\[ g_{\varphi\varphi} = r^2 \]
\[ g_{r\vartheta} = g_{r\varphi} = g_{\vartheta\varphi} = 0 \]

which in the matrix form are written as follows

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & r^2 \sin^2 \varphi & 0 \\
0 & 0 & r^2 \\
\end{pmatrix}
\] (1.100)

The contravariant form of the metric tensor is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{r^2 \sin^2 \varphi} & 0 \\
0 & 0 & \frac{1}{r^2} \\
\end{pmatrix}
\] (1.101)

Using equation (1.85), the Christoffel symbols are

\[
\Gamma^r_{\varphi\varphi} = -r \quad \Gamma^r_{\vartheta\vartheta} = -r \sin^2 \varphi
\]
\[
\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{1}{r} \quad \Gamma^\varphi_{\vartheta\vartheta} = - \sin \varphi \cos \varphi
\]
\[
\Gamma^\vartheta_{r\vartheta} = \Gamma^\vartheta_{\vartheta r} = \frac{1}{r} \quad \Gamma^\vartheta_{\varphi\vartheta} = \Gamma^\varphi_{\vartheta\varphi} = \frac{\cos \varphi}{\sin \varphi}
\]
1.4.4 Volumes and the vector product

In the three-dimensional Euclidean space a volume element $\eta$ is defined as a three-linear form such as

$$\eta := \vec{E} \times \vec{E} \times \vec{E} \rightarrow \mathbb{R}, \quad (\vec{v}_1, \vec{v}_2, \vec{v}_3) \mapsto \eta (\vec{v}_1, \vec{v}_2, \vec{v}_3) \in \mathbb{R} \quad (1.102)$$

When the set of three vectors forms a basis $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$, for any other basis $\{\vec{b}'_1, \vec{b}'_2, \vec{b}'_3\}$, the volume element changes with the following expression

$$\eta (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3) = |a'_{ij}| \eta (\vec{b}_1, \vec{b}_2, \vec{b}_3) \quad (1.104)$$

where $|a'_{ij}|$ is the determinant of the endomorphism for basis changing already seen in equation (1.18).

The application $\eta$ can be expressed by a third order skew-symmetric tensor $\eta_{ijk}$ with the following properties. If two of the subscripts $\{i, j, k\}$ equal each other the volume element vanishes. Any odd permutation of the subscripts changes the sign of the element, any even permutation of the subscripts does not alter the volume element.

For a Cartesian system of coordinates we shall denote the volume form by $\epsilon_{ijk}$ and the above properties become clearer in the following scheme

$$\epsilon_{ijk} = \epsilon^{ijk} = \begin{cases} 0 \text{ when any two of the indices are equal;} \\ 1 \text{ when } i, j, k \text{ is an even permutation of the numbers } 1,2,3; \\ -1 \text{ when } i, j, k \text{ is an odd permutation of the numbers } 1,2,3; \end{cases}$$

that means, for example

$$\epsilon^{112} = \epsilon^{122} = \epsilon^{222} = 0$$

$$\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = \epsilon^{321} = \epsilon^{213} = \epsilon^{231} = 1$$

$$\epsilon^{132} = \epsilon^{321} = \epsilon^{321} = \epsilon^{213} = \epsilon^{213} = -1$$

In addition, the operator $\epsilon_{ijk}$ satisfies the following identity

$$\epsilon_{ijk} \epsilon_{ilh} = \delta_{jl} \delta_{kh} - \delta_{jh} \delta_{kl} \quad (1.105)$$

Suppose that $\{\vec{\partial}_i, \vec{\partial}_j, \vec{\partial}_k\}$ is a basis related to a curvilinear coordinate system, we want to evaluate the volume element in this
system through the volume element expressed in the Cartesian axes.

\[ \eta(\bar{\partial}_i, \bar{\partial}_j, \bar{\partial}_k) = \eta \left( \frac{\partial x^r}{\partial x_i} \bar{\partial}_r, \frac{\partial x^s}{\partial x_j} \bar{\partial}_s, \frac{\partial x^t}{\partial x_k} \bar{\partial}_t \right) = \frac{\partial x^r}{\partial x_i} \frac{\partial x^s}{\partial x_j} \frac{\partial x^t}{\partial x_k} \epsilon_{rst} = \eta_{ijk} \]  

(1.106)

which, by means of equation (1.104), we find that the coefficient of the volume element in the Cartesian coordinates \( \epsilon \) on the left–hand side of the latter equation equals the determinant of the endomorphism of the basis changing, namely

\[ \frac{\partial x^p}{\partial x_i} \frac{\partial x^q}{\partial x_j} \frac{\partial x^t}{\partial x_k} = |a^p_m| = \det \left( \frac{\partial x^p}{\partial x_m} \right) \]  

(1.107)

It is easy to prove that the above determinant is the square root of the determinant of the metric tensor related to the generic coordinate system \( \{ x^i \}, i = 1, 2, 3 \). So we have

\[ \eta(\bar{\partial}_i, \bar{\partial}_j, \bar{\partial}_k) = \sqrt{|g_{pq}|} \epsilon_{ijk} \]  

(1.108)

and in the same way the following contravariant expression can be derived

\[ \tilde{\eta}(d^i, d^j, d^k) = \sqrt{|g^{pq}|} \epsilon^{ijk} \]  

(1.109)

where \( |g_{pq}| = \det (g_{pq}) \) and \( |g^{pq}| = \frac{1}{\det(g_{pq})} = \det (g^{pq}) \).

The skew–symmetric tensor \( \eta \) defines the vector product as follows

\[ \bar{u} \times \bar{v} = u^i \bar{\partial}_i \times v^j \bar{\partial}_j = u^i v^j \eta_{ijk} \]  

(1.110)

and also

\[ u \times v = u_i d^i \times v_j d^j = u_i v_j \eta^{ijk} \bar{\partial}_k \]  

(1.111)

We can use the tensor \( \eta \) to compute infinitesimal volume, area and line elements. Let us begin putting the infinitesimal vector along the \( j \)–th coordinate curve as follows

\[ d\bar{l}_j = dx^j \bar{\partial}_j \]  

(1.112)

so that the infinitesimal volume is given by

\[ dV = \eta(d\bar{l}_1, d\bar{l}_2, d\bar{l}_3) \]  

(1.113)
hence
\[ dV = \eta \left( \bar{\partial}_1, \bar{\partial}_1, \bar{\partial}_1 \right) dx^1 dx^2 dx^3 = \sqrt{g} \epsilon_{123} dx^1 dx^2 dx^3 = \sqrt{g} dx^1 dx^2 dx^3 \] (1.114)

The above expression for the volume element can also be written as
\[ dV = [d\bar{l}_1 \times d\bar{l}_2] \cdot d\bar{l}_3 \] (1.115)
that allows us to attain the same equation expressed in (1.114), indeed we have
\[ [d\bar{l}_1 \times d\bar{l}_2] \cdot d\bar{l}_3 = dx^1 dx^2 \eta_{123} d^3 \left( \bar{\partial}_3 \right) = dx^1 dx^2 dx^3 \sqrt{g} \epsilon_{123} \delta_3^3 = \sqrt{g} dx^1 dx^2 dx^3 \] (1.116)

**Infinitesimal area element**

Taken two infinitesimal vectors along two coordinate curves respectively, the infinitesimal area normal to the vector along the third coordinate curve is given by
\[ dA_3 = |d\bar{l}_1 \times d\bar{l}_2| = \eta_{123} |d^3| dx^1 dx^2 = \sqrt{g} \sqrt{d^3} \cdot d^3 dx^1 dx^2 = \sqrt{g g^{33}} dx^1 dx^2 \] (1.117)
and it is easy to obtain the general expression for any area element
\[ dA_i = \sqrt{g g^{ij}} dx^j dx^k \] (1.118)
where \( i \) is not summed and \( i \neq j \neq k \).

**Infinitesimal line element**

A generic infinitesimal line element \( dl^2 \) is defined by
\[ dl^2 = |d\bar{l}|^2 = d\bar{l} \cdot d\bar{l} = dx^i \bar{\partial}_i \cdot dx^j \bar{\partial}_j = dx^i dx^j g_{ij} \] (1.119)
whereas, a line element taken along the \( i-th \) coordinate curve can be represented by the vector
\[ d\bar{l}_i = dx^i \bar{\partial}_i \quad (i \text{ not summed}) \] (1.120)
and it measures
\[ \sqrt{g} (d\bar{l}_i, d\bar{l}_i) = g_{ii} dx^i \] (1.121)
1.5 Covariant differentiation

In this section we shall briefly introduce some notions concerning
the derivatives of objects so far discussed, i.e. vectors and tensors. In
order to differentiate these fields the concept of manifold is required.
However, in this context it will be restricted to a rough and informal
description.

A manifold is an abstract space locally Euclidean so that, for
each point belonging to the manifold, there is a neighborhood that
can be described as the Euclidean vector space. When we deal with
manifolds, the intuitive idea of vectors obtained by simply subtract-
ing two points in the affine space might no longer be valid. Keep
in mind, for instance, a curved surface \( Q \in E \), i.e. a two dimen-
sional manifold, and try to define a vector entirely belonging to the
surface by subtracting two points. It is easy to see that the vector
cannot belong to the curved surface \( Q \).

For this reason we need an additional space named tangent space
\( T\bar{E} \) that allows us to extend the concept of vector spaces so far
discussed to manifolds. The tangent space is a Euclidean vector
space consisting of the tangent vectors of the curves through the
point of the manifold itself.

In order to use tools for computing volume, area and line el-
ements, i.e. to define the metric tensor, we shall suppose that we
always deal with differentiable Reimannian manifolds. For a formal
mathematical definition see [5].

Given a general coordinate system \( X = \{x^i\}, i = 1, 2, 3 \), let
\( \bar{u} \) be a vector field \( \bar{u} : E \to T\bar{E} \) and \( \tau : E \to \otimes^k T\bar{E} \) a \( k \)-order
contravariant tensor, we define the covariant derivative \( \nabla_{\bar{u}} \tau \) of the
field \( \tau \) with respect to the field \( \bar{u} \) as

\[
\nabla_{\bar{u}} \tau = u^j \left( \partial_j \tau_{i_1 \ldots i_k} + \Gamma_{jh}^{i_1} \tau_{h i_2 \ldots i_k} + \ldots + \Gamma_{j h}^{i_k} \tau_{i_1 \ldots i_k - 1 h} \right) \bar{\partial}_{i_1} \otimes \ldots \otimes \bar{\partial}_{i_k} \tag{1.122}
\]

Analogously, for a \( k \)-order covariant tensor \( \tau : E \to \otimes^k T\bar{E}^* \) the
covariant derivative becomes

\[
\nabla_{\bar{u}} \tau = u^j \left( \partial_j \tau_{i_1 \ldots i_k} - \Gamma_{ji_1}^h \tau_{h i_2 \ldots i_k} - \ldots - \Gamma_{ji_k}^h \tau_{i_1 \ldots i_k - 1 h} \right) d^{i_1} \otimes \ldots \otimes d^{i_k} \tag{1.123}
\]

where \( T\bar{E}^* \) is the cotangent space, namely the space that contains
the dual forms related to the vectors belonging to \( T\bar{E} \).

The above expressions are presented only for the sake of com-
pleteness, while, the covariant derivative of vector fields and second
order tensors, will be often used in the mechanics of shell continua.

In fact, for a second order covariant tensor \( \tau = \tau_{hk} \partial_h \otimes \partial_k \) the derivative is

\[
\nabla \bar{u} \tau = u^j \left( \partial_j \tau_{hk} + \Gamma^h_{jt} \tau_{tk} + \Gamma^k_{jt} \tau_{ht} \right) \bar{\partial}_h \otimes \bar{\partial}_k \tag{1.124}
\]

while, for covariant tensors \( \tau = \tau_{hk} \bar{d}^h \otimes d^k \) the derivative becomes

\[
\nabla \bar{u} \tau = u^j \left( \partial_j \tau_{hk} - \Gamma^t_{jh} \tau_{tk} - \Gamma^t_{jk} \tau_{ht} \right) \bar{d}^h \otimes d^k \tag{1.125}
\]

and for a mixed tensor \( \tau = \tau_{h}^{\ k} d^k \otimes \bar{\partial}_h \) the derivative is

\[
\nabla \bar{u} \tau = u^j \left( \partial_j \tau_{h}^{\ k} + \Gamma^h_{jt} \tau_{t}^{\ k} - \Gamma^t_{jk} \tau_{h}^{\ t} \right) d^k \otimes \bar{\partial}_h \tag{1.126}
\]

Finally, for a vector field we have

\[
\nabla \bar{u} \bar{v} = u^j \left( \partial_j v^i + \Gamma^i_{jh} v^h \right) \bar{\partial}_i \tag{1.127}
\]

and for the dual form

\[
\nabla \bar{u} \bar{v} = u^j \left( \partial_j v_i - \Gamma^h_{ij} v_h \right) d^i \tag{1.128}
\]

### 1.5.1 Grad, div, curl and Laplace’s operator

**Gradient.** Consider a scalar field \( f : E \to \mathbb{R} \), we define the gradient of \( f \) as the vector

\[
\text{grad} \ f = g^{ij} \frac{\partial f}{\partial x^j} \bar{\partial}_i \tag{1.129}
\]

In a Cartesian coordinate system the above operator simplifies in the following expression

\[
\text{grad} \ f = \frac{\partial f}{\partial x^i} \bar{e}_i \tag{1.130}
\]

**Divergence.** We define the divergence of a vector field \( \bar{v} \) as the following scalar

\[
\text{div} \ \bar{v} = \text{tr} \ (\nabla \bar{v}) = v^i_{\ i} + \Gamma^i_{ij} \nu^j \tag{1.131}
\]

In a Cartesian coordinate system the divergence is written as

\[
\text{div} \ \bar{v} = \text{tr} \ (\nabla \bar{v}) = v^i_{\ i} \tag{1.132}
\]
Curl. For the sake of simplicity, to define this operator let us denote by \( \nabla \) a symbolic operator defined as \( \nabla = \frac{\partial}{\partial x^i} d^i \). Now the curl of a vector field \( \vec{v} \) can be defined as the following vector

\[
\text{curl } \vec{v} = \nabla \times g^i (\vec{v}) = \frac{\partial}{\partial x^i} d^i \times v_j g^j = d^i \times \frac{\partial}{\partial x^i} (v_j g^j) = \eta^{ijk} v_{ji} \tilde{\partial}_k
\]

where \( \eta^{ijk} \) is the skew-symmetric tensor related to the vector product (1.111) and \( v_{ij} \) stands for the covariant derivative \( v_{i|j} = v_{ij} - \Gamma^h_{ij} v_h \).

Hence, for a rectangular coordinate system, the curl assumes the straightforward expression

\[
\text{curl } \vec{v} = \nabla \times \vec{v} = v_{ji} \epsilon^{ijk} \hat{e}_k = \vec{\omega}
\]

where \( \vec{\omega} = \omega_k \hat{e}_k \) and \( \omega_k = v_{ji} \epsilon^{ijk} \). Expanding the latter expression leads to the following equivalent form

\[
\text{curl } \vec{v} = \det \begin{pmatrix}
\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\
\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\
v_1 & v_2 & v_3 
\end{pmatrix}
\]

Laplace’s operator. We define the Laplace operator of a scalar field \( f \) the following scalar

\[
\nabla^2 f = g^{ij} \left( \partial_i \partial_j f - \Gamma^h_{ij} \partial_h f \right)
\]

In a rectangular Cartesian coordinate system the Laplacian is written as

\[
\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}
\]
and we point out that vectors (forms) forming the covariant (contravariant) basis \( \{ \bar{\partial}_i \} (\{ \bar{d}^i \}) \) are not dimensionless. Hence, the vector components do not represent a physical quantity, even though their geometric properties are correct. So, in order to give vector components a physical meaning a normalization of the basis is required. To this end we introduce the so called physical basis \( \{ \bar{\partial}_{<i>} \} \) such as

\[
\bar{v} = v^{<i>} \bar{\partial}_{<i>} \tag{1.140}
\]

Next we normalize the covariant basis as follows

\[
\bar{\partial}_{<i>} = \frac{\bar{\partial}_i}{|\bar{\partial}_i|} = \frac{\bar{\partial}_i}{\sqrt{g_{ii}}} \quad (i \text{ not summed}) \tag{1.141}
\]

which, replaced into equation (1.138), allows us to define the physical components of \( \bar{v} \) as follows

\[
v^{<i>} = \sqrt{g_{ii}} v^i \quad (i \text{ not summed}) \tag{1.142}
\]

On the other hand for the dual basis we have

\[
d^{<i>} = \frac{\bar{d}^i}{\sqrt{g^{ii}}} \quad (i \text{ not summed})
\]

\[
v_{<i>} = v_i \sqrt{g^{ii}} \quad (i \text{ not summed})
\]

As an example, in the following we present the expressions of the differential operators discussed in section 1.5.1 for a cylindrical coordinate system.

- **Gradient**

\[
\text{grad } f = g^{\rho \rho} \frac{\partial f}{\partial x^\rho} \bar{\partial}_\rho + g^{\rho \theta} \frac{\partial f}{\partial x^\theta} \bar{\partial}_\theta + g^{zz} \frac{\partial f}{\partial x^z} \bar{\partial}_z = \\
= \frac{\partial f}{\partial x^\rho} \bar{\partial}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial x^\theta} \bar{\partial}_\theta + \frac{\partial f}{\partial x^z} \bar{\partial}_z = \\
= \frac{\partial f}{\partial x^\rho} \bar{\partial}_{<\rho>} + \frac{1}{\rho} \frac{\partial f}{\partial x^\theta} \bar{\partial}_{<\theta>} + \frac{\partial f}{\partial x^z} \bar{\partial}_{<z>}
\]

- **Divergence**

\[
\text{div } \bar{v} = \text{tr } (\nabla \bar{v}) = v^\rho_{,\rho} + v^\theta_{,\theta} + v^z_{,z} + \frac{1}{\rho} v^\rho = \\
= \frac{1}{\rho} \left( v^{<\rho>} + v^{<\theta>} \right) + v^{<\rho>}
\]
\[ \text{Curl} \]
\[ \text{curl } \vec{v} = \eta^{11k} v_{1|1} \vec{\partial}_k + \eta^{12k} v_{2|1} \vec{\partial}_k + \eta^{13k} v_{3|1} \vec{\partial}_k = 0 \]
\[ \eta^{21k} v_{1|2} \vec{\partial}_k + \eta^{22k} v_{2|2} \vec{\partial}_k + \eta^{23k} v_{3|2} \vec{\partial}_k = 0 \]
\[ \eta^{31k} v_{1|3} \vec{\partial}_k + \eta^{32k} v_{2|3} \vec{\partial}_k + \eta^{33k} v_{3|3} \vec{\partial}_k = 0 \]
\[ = \sqrt{|g^{ij}|} \left( (v_{3|2} - v_{2|3}) \vec{\partial}_1 + (v_{1|3} - v_{3|1}) \vec{\partial}_2 + (v_{2|1} - v_{1|2}) \vec{\partial}_3 \right) \]
which by making use of the cylindrical notation as stated in section 1.4.3 (i.e. 1 = \( \rho \), 2 = \( \vartheta \), 3 = \( z \)), taking into account that \( v_{ij} = v_{i,j} \) due to the symmetry of Christoffel symbols in equation (1.98) and considering the physical components, allows the above expression to become
\[ \text{curl } \vec{v} = \left( \frac{1}{\rho} v_{<z>,\vartheta} - v_{<\vartheta>,z} \right) \vec{\partial}_{<\rho>} + \]
\[ + (v_{<\rho>,z} - v_{<z>,\rho}) \vec{\partial}_{<\vartheta>} + \frac{1}{\rho} (v_{<\vartheta>,\rho} - v_{<\rho>,\vartheta}) \vec{\partial}_{<z>} \]

\[ \text{Laplacian} \]
\[ \nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \vartheta^2} + \frac{\partial^2 f}{\partial z^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} \]

Note that in the latter expression no normalization has been used.

**The divergence theorem**

Consider a generic region \( \mathcal{V} \subset E \) bounded by the smooth closed surface \( \mathcal{S} \). Given a continuously differentiable vector field \( \vec{v} \in \mathcal{V} \), we have
\[ \int_{\mathcal{V}} \text{div } \vec{v} \, d\mathcal{V} = \int_{\mathcal{S}} \vec{v} \cdot \vec{n} \, dS \tag{1.143} \]
where \( \vec{n} \) is the outward pointing unit normal vector of the boundary \( \mathcal{S} \).
In components the above theorem becomes
\[
\int_{V} \left( v_{j}^{j} + \Gamma_{jh}^{j} v^{h} \right) \, dv = \int_{S} v^{i} n_{i} \, dS \tag{1.144}
\]

The divergence theorem holds for tensor fields. For a mixed II-order tensor \( \tau = \tau_{k}^{h} (d^{k} \otimes \bar{\partial}_{h}) \), for example, the theorem states
\[
\int_{V} \text{div} \, \tau \, dv = \int_{S} \tau (n) \, dS \tag{1.145}
\]
where the \( k \)-th covariant component is
\[
\int_{V} \left( \tau_{k,h}^{h} + \Gamma_{ht}^{h} \tau_{k}^{t} - \Gamma_{hk}^{t} \tau_{t}^{h} \right) \, dv = \int_{S} \tau_{k}^{h} n_{h} \, dS \tag{1.146}
\]

While for a II-order contravariant tensor \( \tau = \tau^{hk} (\bar{\partial}_{h} \otimes \bar{\partial}_{k}) \) it becomes
\[
\int_{V} \left( \tau_{h,k}^{hk} + \Gamma_{ht}^{h} \tau_{t,k}^{k} + \Gamma_{ht}^{k} \tau_{ht}^{h} \right) \, dv = \int_{S} \tau^{hk} n_{h} \, dS \tag{1.147}
\]

### 1.6 Affine space

Here we shortly introduce the notion of affine space.

Let \( \bar{E} \) be a \( n \)-dimensional vector space. We define the affine space associated to \( \bar{E} \) the set of points \( E \) equipped with the translation \(+\), such as
\[
+: E \times \bar{E} \to E : (p, \bar{u}) \mapsto p + \bar{u} = p' \in E \tag{1.148}
\]
where \( \bar{u} = (p' - p) \in \bar{E} \) represents a free vector, while the pairs \((p, \bar{u})\) form applied vectors.

#### 1.6.1 Free and applied vectors

This section is restricted to the geometrical interpretation of vectors belonging to the Euclidean space and expressed through the rectangular coordinate system. So that we have
\[
g_{ij} = g^{ij} = g^{i}_{j} = \delta_{ij} \tag{1.149}
\]
and
\[
\eta_{ijk} = \epsilon_{ijk} \tag{1.150}
\]
From a geometric point of view an applied vector is represented by a line segment $\overrightarrow{AB}$ from point $A$ to point $B$, where, with respect to equation (1.148), $A = p$ and $B = p'$. If $B$ is moved to the position $C$, then the whole translation from $A$ to $C$ represents the sum of the partial translations $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

Putting $\overrightarrow{AB} = \overrightarrow{a}$ and $\overrightarrow{BC} = \overrightarrow{b}$ we notice that if they were applied in the same point, see figure 1.4, then a practical rule can be used to carry out the addition $\overrightarrow{a} + \overrightarrow{b}$. It consists in moving the vector $\overrightarrow{b}$, in such a way to be kept parallel to itself, into a new position so that its starting point coincides with the ending point of $\overrightarrow{a}$. Thus, the line segment from $A$ to the end point of $\overrightarrow{b}$ (in the new position) represents the addition $\overrightarrow{a} + \overrightarrow{b}$. See figure 1.4. This rule is known as parallelogram rule because $\overrightarrow{a}$ and $\overrightarrow{b}$ form the sides of a parallelogram and $\overrightarrow{a} + \overrightarrow{b}$ is one of the diagonals.

![Figure 1.4: Addition of two applied vectors.](image)

The subtraction of two vectors applied in the same point can be seen as $\overrightarrow{c} = \overrightarrow{a} + (-\overrightarrow{b})$ and so it is carried out by means of the procedure described for the addition. The vector $\overrightarrow{c} = \overrightarrow{a} - \overrightarrow{b}$ will be given by the line joining the starting point of $\overrightarrow{a}$ to the end point of $-\overrightarrow{b}$. See figure 1.5.

The addition of two applied vectors has the following properties

1. $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} + \overrightarrow{a}$;
2. $(\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c})$;
3. $(\lambda + \mu) \overrightarrow{a} = \lambda \overrightarrow{a} + \mu \overrightarrow{a}$;
4. $\lambda (\mu \overrightarrow{a}) = (\lambda \mu) \overrightarrow{a}$;
5. $\lambda (\overrightarrow{a} + \overrightarrow{b}) = \lambda \overrightarrow{a} + \lambda \overrightarrow{b}$;
Figure 1.5: Subtraction of two applied vectors.

where $\vec{a}, \vec{b}, \vec{c}$ are applied vectors in $\vec{E}$ and $\lambda, \mu \in \mathbb{R}$.

For an applied vector it is possible to define norm, direction, sign:

**norm (modulus or length)**: is the length, measured by a fixed unit system, of the line segment $\overrightarrow{AB}$;

**direction**: is the direction of the line passing through $A$ and $B$;

**sign**: specifies the sign, i.e. $\overrightarrow{AB} = -\overrightarrow{BA}$.

From the preceding discussion about the metric tensor it is known that the length (modulus) of a vector $\overrightarrow{a} (= \overrightarrow{AB})$ is the square root of the scalar product by itself

$$\| \overrightarrow{a} \| = \sqrt{g(\overrightarrow{a}, \overrightarrow{a})} = \sqrt{\overrightarrow{a} \cdot \overrightarrow{a}} \quad (1.151)$$

Recalling that the metric tensor is a bilinear symmetric positive definite form, the following properties can be derived

1. $\overrightarrow{a} \cdot \overrightarrow{a} = \| \overrightarrow{a} \|^2 > 0$ se $\overrightarrow{a} \neq \overrightarrow{0}$;
2. $\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{b} \cdot \overrightarrow{a}$;
3. $\overrightarrow{c} \cdot (\overrightarrow{a} + \overrightarrow{b}) = \overrightarrow{c} \cdot \overrightarrow{a} + \overrightarrow{c} \cdot \overrightarrow{b}$;
4. $\lambda (\overrightarrow{a} \overrightarrow{b}) = (\lambda \overrightarrow{a}) \cdot \overrightarrow{b} = \overrightarrow{a} \cdot (\lambda \overrightarrow{b})$;

The cross product of two applied vectors $\overrightarrow{a}, \overrightarrow{b} \in \vec{E}$ in a Cartesian coordinate system is carried out by using the general rule given in equation (1.110), so that

$$\overrightarrow{\omega} = \overrightarrow{a} \times \overrightarrow{b} \quad \overrightarrow{\omega} \in \vec{V} \quad (1.152)$$

where
modulus: $\| \vec{w} \| = \| \vec{a} \| \| \vec{b} \| \sin \theta$, where $\theta$ denotes the angle between $\vec{a}$ and $\vec{b}$;

direction: normal to the plane to which $\vec{a}$ and $\vec{b}$ belong;

sign: follows the right hand rule.

Moreover, by virtue of the skew-symmetric tensor $\epsilon_{ijk}$, the vector product vanishes when either one of the two vectors vanishes or when the two vectors are parallel. See figure 1.6.

\[ \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}; \]
\[ (\lambda \vec{a} + \mu \vec{b}) \times \vec{c} = \lambda (\vec{a} \times \vec{c}) + \mu (\vec{b} \times \vec{c}). \]

1.7 Surfaces

Let $E$ be the affine Euclidean space. The submanifold $Q \subset E$ is a surface if $\dim Q = 2$.

Suppose $Q \subset E$ is a surface which can be described by an induced coordinate system of dimension $q = m - k$, where $m$ is the dimension of $E$ and $k$ denotes the number of constraints (codimension of $Q$). Since $Q$ is a surface we have $m = 3$, $k = 1$, $q = 2$. The induced coordinate system is given by

\[ X^\dagger : Q \to \mathbb{R}^q : p \mapsto x^\alpha(p) \]
From now on the quantities living on $Q$ will be distinguished by the symbol $\dagger$ and the components will be written using superscripts and subscripts, running from 1 to 2, in Greek letters. The Latin indices will denote components of quantities that are applied on $Q$ but lie out, namely belonging to the vector space $T_Q E$.

The unit normal vector is defined as follows

$$\bar{n} : Q \to T_Q \perp \text{ so that } g(\bar{n}, \bar{n}) = 1.$$  \hfill (1.154)

where $g$ is the metric tensor defined on $TE$ and $T_Q \perp$ is the orthogonal space.

Analogously, on the surface $Q$ we can define the the induced metric as

$$g\dagger : TQ \times TQ \to \mathbb{R} \text{ that in components\textsuperscript{3} becomes}$$

$$g\dagger = g_{\alpha\beta} d^\alpha \otimes d^\beta$$

Given two vectorial fields $\bar{u} : Q \to TQ$ and $\bar{v} : Q \to TQ$, the covariant derivative of $\bar{v}$ with respect to $\bar{u}$ can be split as follows

$$\nabla_{\bar{u}} \bar{v} = \nabla_{\bar{u}} \parallel \bar{v} + \nabla_{\bar{u}} \perp \bar{v} \hfill (1.155)$$

where

$$\nabla \parallel : TQ \times TQ \to TQ \hfill (1.156)$$

$$\nabla \perp : TQ \times TQ \to T_Q \perp \hfill (1.157)$$

The application $\nabla \parallel$ is called second fundamental form of the surface. For further details see [5] and [3].

We now define the Weingarten\textsuperscript{4} map $L$ as the following endomorphism

$$L := \nabla \bar{n} : TQ \to TQ : \bar{u} \mapsto \nabla_{\bar{u}} \bar{n} \hfill (1.158)$$

\textsuperscript{3}In some books the covariant components of the metric tensor $g\dagger$ are also denoted as $g_{11} = E, \ g_{12} = F, \ g_{22} = G$.

\textsuperscript{4}Julius Weingarten (March 2, 1836 Berlin - June 16, 1910 Freiburg) was a German mathematician.

Source: http://www-history.mcs.st-andrews.ac.uk/.
In addition to that, we define the total curvature (Gauss curvature) $K$ and the mean curvature $H$ of a surface $Q$ as follows

$$K := \det L : Q \to \mathbb{R}$$

$$H := \text{tr} L : Q \to \mathbb{R}$$

Finally, eigenvalues of $L$ are defined principal curvatures. See [5].

Let $L$ be the second order covariant tensor related to the Weingarten endomorphism $L$ by the metric tensor $g^\flat$, so that

$$L := \nabla_\bar{u} : T^*Q \times T^*Q \to \mathbb{R} : (\bar{u}, \bar{v}) \mapsto g(L(\bar{u}), \bar{v}) = \nabla_\bar{u} \bar{n} \cdot \bar{v}$$

where $\bar{u} = g^\flat (\bar{n})$.

The following differentiation

$$0 = \nabla_{\bar{u}} (g(\bar{v}, \bar{n})) = g(\nabla_{\bar{u}} \bar{v}, \bar{n}) + g(\bar{v}, \nabla_{\bar{u}} \bar{n}) \Rightarrow$$

$$g(\nabla_{\bar{u}} \bar{v}, \bar{n}) = -g(\bar{v}, \nabla_{\bar{u}} \bar{n})$$

proves that the scalar quantity $L(\bar{u}, \bar{v})$ represents the normal component to the surface $Q$ of the covariant derivative, namely

$$\nabla_{\bar{u}} \bar{v} = \nabla_{\bar{u}}^{\|} \bar{v} - L(\bar{u}, \bar{v}) \bar{n}$$

Dealing with mechanics of shell continuaums, equation (1.164) will be often used. Hence, in the following we expand its expression in components.

Suppose $\{\bar{\partial}_\alpha\}$, $\alpha = 1, 2$ is a basis related to the induced coordinate system describing the surface, we have

$$\nabla_{\bar{\partial}_\beta} \bar{\partial}_\alpha = \nabla_{\bar{\partial}_\beta}^{\dagger} \bar{\partial}_\alpha - L(\bar{\partial}_\beta, \bar{\partial}_\alpha) \bar{n}$$

and for both right hand terms we have, respectively

$$\nabla_{\bar{\partial}_\beta}^{\dagger} \bar{\partial}_\alpha = d^\gamma (\bar{\partial}_\beta) \left( \partial_\gamma (d^\omega (\bar{\partial}_\alpha)) + \Gamma^\omega_\beta_\gamma d^\lambda (\bar{\partial}_\alpha) \right) \bar{\partial}_\omega$$

$$= \delta^\gamma_\beta \left( \Gamma^\omega_\beta_\gamma \bar{\partial}_\alpha \right) \bar{\partial}_\omega = \Gamma^\omega_\beta_\gamma \bar{\partial}_\omega$$

$$L(\bar{\partial}_\beta, \bar{\partial}_\alpha) = (L(\bar{\partial}_\beta) \cdot \bar{\partial}_\alpha) = \nabla_{\bar{\partial}_\beta} \bar{n} \cdot \bar{\partial}_\alpha$$

$$= L_{\beta_\omega}^\omega \bar{\partial}_\omega \cdot \bar{\partial}_\alpha = L_{\beta_\omega}^\omega g_{\omega\alpha} = L_{\beta_\alpha}$$
Finally, equation (1.165) in components becomes
\[ \nabla_\beta \bar{\partial}_\alpha = \Gamma^\omega_\beta_\alpha \bar{\partial}_\omega - L_\beta_\alpha \bar{n} \] (1.170)

Note that in the remainder of this book, for the sake of brevity, we will use \( \nabla_\beta \cdot \) instead of \( \nabla_\partial_\beta \cdot \).

Analogously, for an element of the contravariant basis, recalling the general equation for covariant derivatives, and considering the above Gauss splitting, we have the following expression
\[ \nabla_\beta d^\alpha = -\Gamma^\alpha_\beta_\gamma d^\gamma - L^\alpha_\beta \bar{n} \] (1.171)

Often, for instance in the case of shell theory, we will deal with vector fields that do not belong to the tangent space, so it is useful to present an example of derivative of vectors applied in \( Q \) but lying out of the tangent space. Namely, suppose that \( \bar{v} \in T\bar{Q}E \). We can decompose the field \( \bar{v} \) into the tangent and orthogonal component as follows
\[ \bar{v} = \bar{v}^\parallel + \bar{v}^\perp \] (1.172)
that in components is written as
\[ \bar{v} = v^\alpha \bar{\partial}_\alpha + v^\xi \bar{n} \] (1.173)

Hence, given \( \bar{u} \in \bar{T}Q \) the derivative of \( \bar{v} \) with respect to \( \bar{u} \) is
\[ \nabla_{\bar{u}} \bar{v} = \nabla_{\bar{u}} \bar{v}^\parallel + \nabla_{\bar{u}} \bar{v}^\perp = \nabla^\dagger_{\bar{u}} \bar{v}^\parallel - L(\bar{u}, \bar{v}^\parallel) \bar{n} + \nabla_{\bar{u}} \bar{v}^\perp \] (1.174)
that in components turns into
\[ \nabla_{\bar{u}} \bar{v} = u^\beta \left( \partial_\beta v^\alpha + \Gamma^\alpha_\beta_\gamma v^\gamma + v^\xi L^\alpha_\beta \right) \bar{\partial}_\alpha + u^\beta \left( v^\xi - L^\alpha_\beta v^\alpha \right) \bar{n} \] (1.175)

In the same way, the dual form \( \bar{v} \in T^*_\bar{Q}E \) can be differentiated as follows
\[ \nabla_{\bar{u}} \bar{v} = \nabla_{\bar{u}} \bar{v}^\parallel + \nabla_{\bar{u}} \bar{v}^\perp = \nabla^\dagger_{\bar{u}} \bar{v}^\parallel - L(\bar{u}, \bar{v}^\parallel) \bar{n} + \nabla_{\bar{u}} \bar{v}^\perp \] (1.176)
that in components becomes
\[ \nabla_{\bar{u}} \bar{v} = u^\beta \left( \partial_\beta v^\alpha - \Gamma^\gamma_\alpha_\beta v^\gamma + v^\xi L^\alpha_\beta \right) d^\gamma + u^\beta \left( v^\xi - L^\alpha_\beta v^\alpha \right) \bar{n} \] (1.177)

Examples of surfaces will be provided in appendix A, where, within the application of the shell theory, the above results will be applied to some well known geometries.
Chapter 2
Analysis of strain

This chapter is devoted to the classical strain theory for deformable continua. In order to offer a comprehensive approach, the first part will be treated in curvilinear coordinates, then results in Cartesian coordinates will be obtained as a special case.

2.1 Introduction

Before introducing the definition of strain it is useful to give some preliminary concepts and definitions.

Let us begin with the definition of body.

A body $C \subset E$ consists of a set of particles embedded in the three-dimensional Euclidean space. Each particle $p \in C$, i.e. a material point, can be put in one-to-one correspondence with a triplet of scalars that univocally determine the position of such a point. Namely, for any point $p$ included in the body there exists a coordinate system $X : C \subset E \rightarrow \mathbb{R}^3$. See also the more general expression (1.60) on section 1.4.

From the notions of body and time we can derive the concept configuration. Configurations are regions $\mathcal{V}$ of the three-dimensional Euclidean space $E$ that can be occupied by the body in a particular instant. Thus we have

$$\mathcal{V} \equiv (C, t) = \{(p, t) | p \in C\} \quad (2.1)$$

where $\mathcal{V}$ is also called a spacial domain for fixed $t$.

It is assumed that:

- **Configurations** are open connected sets or domains in the Euclidean space.

- On varying of the time $t$, the configurations of one and the same body maintain a continuous one-to-one correspondence between different positions of one and the same particle.
2.2 Deformation

Now, beginning with an intuitive statement, we can introduce the definition of strain. When the relative position of two points included in a continuous media is altered, we say that the body is strained. Hence, analysis of strains means to evaluate the change of the relative distance between points; this is also called deformation\(^1\).

2.3 Strain tensor in general coordinates

Let \( \mathcal{V} \) be the region taken by an unstrained state of a body at time \( t \), so that

\[
\mathcal{V} \equiv (\mathcal{C}, t)
\]

and \( \mathcal{V}' \) the configuration of the body in the strained state at instant \( t' \), that is

\[
\mathcal{V}' \equiv (\mathcal{C}, t')
\]

Consider a Cartesian coordinate system equipped with the unit normal basis \( \{\vec{e}_i\} \), so that for any point \( p \) in \( \mathcal{V} \) and \( p' \) in \( \mathcal{V}' \) the positional vectors can be written respectively as

\[
\vec{r} = (p - o) = x^i_c \vec{e}_i
\]

\[
\vec{r}' = (p' - o) = y^i_c \vec{e}_i
\]

We assume that each point in \( \mathcal{V}' \) is related to its original position in \( \mathcal{V} \), and vice versa, by the following relations

\[
y^i_c = y^i_c (x^1_c, x^2_c, x^3_c, t)
\]

\[
x^i_c = x^i_c (y^1_c, y^2_c, y^3_c, t)
\]

In order to avoid penetrations or separations of the material particles it is necessary that the transformation of points in \( \mathcal{V} \) into points in \( \mathcal{V}' \) is one-to-one. Namely, to ensure the existence of the single-valued inverse of equation (2.6) (or (2.7)) it is sufficient to

\(^1\)We know that in nature all materials are deformable, but sometimes we will refer to the abstraction of non-deformable (or rigid) body. This abstraction assumes that for every pair of points belonging to the continuum, the relative distance remains unvaried throughout the history of the motion.
assume that the functions \( y^i_c \) and \( x^i_c \) are continuous and differentiable as many times as required and the Jacobian is greater than zero\(^2\). We write, accordingly

\[
\left| \frac{\partial y^i_c}{\partial x^j_c} \right| > 0
\]

Consider now a generic curvilinear coordinates system \( X = \{x^i\} \) so that

\[
\vec{r} = x^i \vec{\partial}_i = x_i \vec{g}^i
\]

where \( \{\vec{\partial}_i\} \) and \( \{\vec{d}^i\} \) are the covariant and contravariant bases related to the curvilinear system and \( x^i = r^i, \ r_i = x_i \). See figure 2.1.

Points belonging to the initial configuration \( \mathcal{V} \) can also be related to the curvilinear system of coordinates as follows

\[
x^i_c = x^i_c(x^1, x^2, x^3)
\]

where \( x^i_c \) are single-valued and differentiable as many times as required\(^3\).

Moreover, we can use the curvilinear coordinates to describe the body in the strained configuration \( \mathcal{V}' \), so that

\[
y^i_c = y^i_c(x^1, x^2, x^3)
\]

According to section 1.4.2, through the Jacobian matrices, we can compute the metric tensors \( g \) and \( g' \) associated to the curvilinear coordinate system for both configurations, respectively.

For the unstrained configuration the covariant components of the metric tensor are

\[
g_{ij} = \vec{\partial}_i \cdot \vec{\partial}_j = \frac{\partial x^h_c}{\partial x^i_c} \bar{\vec{e}}_h \cdot \frac{\partial x^k_c}{\partial x^j_c} \bar{\vec{e}}_k = \delta_{hk}\]

\[
= \frac{\partial x^h_c}{\partial x^i_c} \frac{\partial x^k_c}{\partial x^j_c} \delta_{hk} = \frac{\partial x^h_c}{\partial x^i_c} \frac{\partial x^h_c}{\partial x^j_c}
\]

\(^2\)The Jacobian of the function \( y^i_c = y^i_c(x^j_c) \) is the determinant of the matrix whose \( i \)-th row lists all the first-order partial derivatives of \( y^i_c \).

\(^3\)With the exception of singular points, curves, surfaces.
while the contravariant components are

\[ g^{ij} = d^i \cdot d^j = \frac{\partial x^i}{\partial x^h_c} e^h \cdot \frac{\partial x^j}{\partial x^k_c} e^k = \]

(2.13)

\[ = \frac{\partial x^i}{\partial x^h_c} \frac{\partial x^j}{\partial x^k_c} \delta^{hk} = \frac{\partial x^i}{\partial x^h_c} \frac{\partial x^j}{\partial x^k_c} \]

(2.14)

and finally the mixed components are

\[ g^i_j = d^i \left( \partial_j \right) = \frac{\partial x^i}{\partial x^h_c} e^h \cdot \frac{\partial x^i}{\partial x^k_c} e^k = \]

(2.15)

\[ = \frac{\partial x^i}{\partial x^h_c} \frac{\partial x^i}{\partial x^k_c} \delta^h_k = \frac{\partial x^i}{\partial x^h_c} \frac{\partial x^i}{\partial x^k_c} \]

(2.16)

For the strained configuration the covariant components of the metric tensor are

\[ g'_{ij} = \bar{d}'_i \cdot \bar{d}'_j = \frac{\partial y^i}{\partial y^h_c} \bar{e}^h \cdot \frac{\partial y^j}{\partial y^k_c} \bar{e}^k = \]

(2.17)

\[ = \frac{\partial y^i}{\partial y^h_c} \frac{\partial y^k}{\partial y^i_c} \delta^{hk} = \frac{\partial y^i}{\partial y^h_c} \frac{\partial y^k}{\partial y^i_c} \]

(2.18)

the contravariant components are

\[ g'^{ij} = \bar{d}^{di} \cdot \bar{d}^{dj} = \frac{\partial x^i}{\partial y^h_c} e^h \cdot \frac{\partial x^j}{\partial y^k_c} e^k = \]

(2.19)

\[ = \frac{\partial x^i}{\partial y^h_c} \frac{\partial x^j}{\partial y^k_c} \delta^{hk} = \frac{\partial x^i}{\partial y^h_c} \frac{\partial x^j}{\partial y^k_c} \]

(2.20)
and finally the mixed components are

\[ g'_{ij} = \frac{\partial x^i}{\partial y^h_c} \frac{\partial y^k_c}{\partial x^j} = \frac{\partial x^i}{\partial y^h_c} \frac{\partial y^k_c}{\partial x^j} \delta^h_k = \partial x^i \partial y^k_c \delta^h_k = \partial x^i \partial y^h_c \delta^k_j \] (2.22)

At the beginning of this chapter we said that the aim of the analysis of strain is to evaluate the change of length between two points in a continuous medium. We are now mathematically able to evaluate this difference

\[ dl'^2 - dl^2 \] (2.23)

where \( dl'^2 = |\bar{\ell}'|^2 \) and \( dl^2 = |\bar{\ell}|^2 \) are the arc lengths of the strained and unstrained states, respectively. Namely, the vector \( d\bar{l} \), joining the points \( p_0 \) and \( p \), during the transformation, is carried into another vector \( d\bar{l}' \). These two vectors differ in direction and magnitude. See figure 2.2.

By using equation (1.119) on page 23, we can write the above line elements with the help of the metric tensors for both configurations as

\[ dl^2 = g_{ij} dx^i dx^j \] (2.24)
\[ dl'^2 = g'_{ij} dx^i dx^j \] (2.25)

then, the difference

\[ dl'^2 - dl^2 = (g'_{ij} - g_{ij}) dx^i dx^j \] (2.26)

We now define a symmetric tensor named the strain tensor, as

\[ \gamma_{ij} = \frac{1}{2} (g'_{ij} - g_{ij}) \] (2.27)

so that

\[ dl'^2 - dl^2 = 2\gamma_{ij} dx^i dx^j \] (2.28)

The strain tensor is obtained by subtracting two bilinear forms so that it is still a bilinear form. Therefore, given two vectors \( \vec{\bar{p}}_0 \) and \( \vec{\bar{q}}_0 \) at a fixed time \( t_0 \) (let us say the initial unstrained state), the strain tensor just measures the difference between the scalar
Figure 2.2: Measure of strain.

The product of the vectors \( \bar{p} \) and \( \bar{q} \) at a generic time \( t \) (that identifies the strained state) and the scalar product at the initial state.

\[
\gamma : E \times E \rightarrow IR \tag{2.29}
\]

\[(\bar{p}, \bar{q}) \mapsto \gamma (\bar{p}, \bar{q}) = g'(\bar{p}, \bar{q}) - g(\bar{p}, \bar{q}) \tag{2.30}\]

so that

\[
\gamma (\bar{p}, \bar{q}) = p^h q^k \gamma_{ij} (\overline{d^i} \otimes \overline{d^j}) (\overline{\partial_h} \partial_k) =
\]

\[= p^h q^k \gamma_{ij} \delta_h^i \delta_k^j = \gamma_{ij} p^i q^j. \tag{2.31}\]

Points in \( \mathcal{V} \) and \( \mathcal{V}' \) are univocally determined by the positional vectors \( \overline{r} \) and \( \overline{r}' \) respectively. With respect to the generic curvilinear coordinate system \( X \) we have

\[
\overline{r} = x^i \overline{\partial_i} \tag{2.32}
\]

\[
\overline{r}' = y^i \overline{\partial_i} \tag{2.33}
\]

hence, the position \( p' \) relative to \( p \) is denoted \( \overline{u} \) and it is called the displacement vector

\[
\overline{u} = \overline{r}' - \overline{r}. \tag{2.34}
\]

Considering now that the basis related to the curvilinear coordinates is given using equation (1.76), we have in the following an
equivalent expression

\[ \bar{\partial}_i = \frac{\partial \bar{r}}{\partial x^i} \]  
(2.35)

\[ \bar{\partial}'_i = \frac{\partial \bar{r}'}{\partial x^i} \]  
(2.36)

and by considering the relation (2.34) it becomes

\[ \bar{\partial}'_i = \bar{r}'_{,i} = \bar{r}_{,i} + \nabla_i \bar{u} \]  
(2.37)

where the comma denotes the partial derivative and \( \nabla \) indicates the covariant derivative.

\[ \gamma_{ij} = \frac{1}{2} \left( \bar{\partial}'_i \cdot \bar{\partial}'_j - \bar{\partial}_i \cdot \bar{\partial}_j \right) = \]  
(2.38)

\[ = \frac{1}{2} \left( (\bar{\partial}_i + \nabla_i \bar{u}) \cdot (\bar{\partial}_j + \nabla_j \bar{u}) - \bar{\partial}_i \cdot \bar{\partial}_j \right) = \]  
(2.39)

\[ = \frac{1}{2} \left( \bar{\partial}_i \cdot \nabla_j \bar{u} + \bar{\partial}_j \cdot \nabla_i \bar{u} + \nabla_i \bar{u} \cdot \nabla_j \bar{u} \right) \]  
(2.40)

In fact, recalling the general expression (1.122) for this differentiation, the above equation turns into

\[ \bar{\partial}'_i = \bar{r}'_{,i} = \bar{\partial}_i + \left( u^m_{,i} + \Gamma_{ih}^m u^h \right) \bar{\partial}_m \]  
(2.41)

where the Christoffel symbols are referred to the metric tensor related to the curvilinear coordinates for the original configuration \( \mathcal{V} \) of the body.

Finally, using the definition of strain tensor, with some calculations we can obtain the expression of the \textit{finite strain tensor} in general coordinates as

\[ \gamma_{ij} = \frac{1}{2} \left( \nabla_j u^i + \nabla_i u^j + \nabla_i u^h \nabla_j u^h \right) \]  
(2.42)

Expanding the above derivatives the strain tensor assumes the following expression

\[ \gamma_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} + 2\Gamma^h_{ji} u^h \right) + \]  

\[ \frac{1}{2} \left( u_{k,i} u^k_{,j} + u_{k,i} \Gamma^k_{js} u^s + u_{k,j} \Gamma^k_{is} u^s + \Gamma^p_{ih} u^p \Gamma^h_{js} u^s \right) \]  
(2.43)
For Cartesian coordinate systems we could repeat exactly the above procedure to obtain the strain tensor, but this is equivalent to putting zero all the Christoffel symbols in equation (2.43). So that for rectangular coordinate systems the strain tensor assumes the following expression

\[ \gamma_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \]  

(2.44)

where we remind the reader again that in the rectangular coordinates the position of the indices does not make any difference because \( g_{ij} = \delta_{ij} \).

### 2.3.1 Examples of strain in Cartesian coordinates

#### Stretching ratio

Let us define the stretching ratio \( \delta_l \) as follows

\[ \delta_l = \frac{dl' - dl}{dl} = \frac{dl'}{dl} - 1 \]  

(2.45)

Namely, suppose we have two points in the unstrained state the difference of which gives a vector \( dl = dx^i \bar{e}_i \). The corresponding vector in the strained state is \( dl' = dx'^i \bar{e}'_i \). Therefore the stretching ratio \( \delta_l \) gives the relative difference between the length of the vectors \( dl \) and \( dl' \).

By means of the definition (2.28) we have

\[ 2 \frac{\gamma_{ij}dx^idx^j}{dx^kdx^k} = \frac{dl'^2}{dl^2} - 1 \]  

(2.46)

then

\[ \delta_l + 1 = \sqrt{1 + 2 \frac{\gamma_{ij}dx^idx^j}{dx^kdx^k}} \]  

(2.47)

so that the stretching ratio can be written as follows

\[ \delta_l = \sqrt{1 + 2 \frac{\gamma_{ij}dx^idx^j}{dx^kdx^k}} - 1 \]  

(2.48)

Considering a simple extension along one of the \( x_i \)-axis we have \( dl = \bar{e}_i \), the stretching turns into

\[ \delta_i = \sqrt{1 + 2 \gamma_{ii}} - 1 \]  

(2.49)
Angular dilatation

Let us consider the vectors $\vec{d}l$ and $\vec{d}s$ at a position $p$ in the unstrained state which are deformed into vectors $\vec{d}l'$ and $\vec{d}s'$, respectively. The difference between the angle amid the deformed vectors and the unstrained vectors is called angular dilatation. For the sake of simplicity, suppose that $\vec{d}l = \vec{e}_1$ and $\vec{d}s = \vec{e}_2$. We define the angular dilatation the following difference

$$\omega_{12} = \frac{\pi}{2} - \varphi'_{12} \quad (2.50)$$

\[\text{Figure 2.3: Angular dilatation.}\]

See figure 2.3.

The scalar product of the strained vectors is

$$d\vec{l}' \cdot d\vec{s}' = dl'ds' \cos \varphi'_{12} \quad (2.51)$$

and the modulus of both strained vectors can be written by means of the preceding result for the linear dilatation

$$d\vec{l}' = (1 + \delta_1) dl = 1 + \delta_1 \quad (2.52)$$
$$d\vec{s}' = (1 + \delta_2) ds = 1 + \delta_2 \quad (2.53)$$

so that equation (2.51) becomes

$$d\vec{l}' \cdot d\vec{s}' = (1 + \delta_1)(1 + \delta_2) \cos \varphi'_{12} \quad (2.54)$$
The left hand term of the latter expression can be written with the help of the strain tensor, so that by recalling equation (2.27) we have

\[ dl^i \cdot ds^j = (\delta_{ij} + 2\gamma_{ij}) dl^i ds^j = 2\gamma_{12} \]  \hspace{1cm} (2.55)

Finally equation (2.54) turns into

\[ 2\gamma_{12} = (1 + \delta_1)(1 + \delta_2) \cos \varphi'_{12} \]  \hspace{1cm} (2.56)

By virtue of the identity \( \sin \omega_{12} = \cos \varphi_{12} \), the angular dilatation becomes

\[ \sin \omega_{12} = \frac{2\gamma_{12}}{(1 + \delta_1)(1 + \delta_2)} \]  \hspace{1cm} (2.57)

and naturally the above formula can be used to compute also the dilatations \( \omega_{23} \) and \( \omega_{31} \).

**Area dilatation**

Vectors \( dl \) and \( ds \) at a position \( p \) in the unstrained state define an area element \( dA \) which is deformed into the area element \( dA' \) defined by the strained vectors \( dl' \) and \( ds' \). We define the *area dilatation ratio* the following coefficient

\[ \alpha = \frac{dA' - dA}{dA} \]  \hspace{1cm} (2.58)

We may suppose for simplicity that \( dl = \bar{e}_1 \) and \( ds = \bar{e}_2 \). See figure 2.4.

![Figure 2.4: Area dilatation.](image)
As well known, we have
\[ dA' = |d\vec{l}' \times \vec{s}'| = dl'ds' \sin \varphi'_{12} \] (2.59)
and, recalling equations (2.52) and (2.53), the latter expression becomes
\[ dA' = dl'ds' \sin \varphi'_{12} = (1 + \delta_1)(1 + \delta_2) \sin \varphi'_{12} \] (2.60)
finally, through the geometrical relation \( \cos \omega_{12} = \sin \varphi_{12} \) it is easy to reach the following expression for the finite area dilatation ratio
\[ \alpha = (1 + \delta_1)(1 + \delta_2) \cos \omega_{12} - 1 \] (2.61)
that can be alternatively written as
\[ \alpha = (1 + \delta_1)(1 + \delta_2) \sqrt{1 - \sin^2 \omega_{12}} - 1 = \]
\[ = \sqrt{(1 + \delta_1)(1 + \delta_2) - 4\gamma^2_{12}} \] (2.62)

Volume dilatation

We define the \textit{volume dilatation ratio} the coefficient
\[ \nu = \frac{dV' - dV}{dV} \] (2.63)

As in the preceding cases, let us suppose that the initial unstrained volume is given by the following unit vectors
\[ dV = [\vec{e}_1 \times \vec{e}_2] \cdot \vec{e}_3 = \epsilon_{123} = 1 \] (2.64)
Thus, the volume dilatation turns into
\[ \nu = dV' - 1 \] (2.65)
For the strained volume we have
\[ dV' = [dl'_1 \times dl'_2] \cdot dl'_3 = \]
\[ = dl'_1 dl'_2 dl'_3 = \]
\[ = (1 + \delta_1)(1 + \delta_2)(1 + \delta_3) \] (2.66)
Finally, the volume dilatation ratio becomes
\[ \nu = (1 + \delta_1)(1 + \delta_3)(1 + \delta_3) - 1 \] (2.69)
2.3.2 Infinitesimal deformations

In the preceding section we have discussed the theory of finite deformations. Now, if all the components of the displacements $\bar{u}$ and the displacement gradient tensor $u_{i,j}$ are very small we may neglect the squares and the product of these quantities in comparison with the first order derivatives themselves. So equation (2.99) becomes

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (2.70)$$

where $\varepsilon$ denotes a symmetric second-order tensor named *infinitesimal strain tensor*.

Explicit compatibility equations

Now we want to know if any state of given strain $\varepsilon_{ij}$ yields a displacement field $u_j$ at every point $p \in V$. To ensure that we have found equations (2.70) and to solve the differential equations system we discard the components of displacements $u_i$ as follows

$$2 \varepsilon_{ij,hk} = u_{i,jhk} + u_{j,ihk} \quad (2.71)$$
$$2 \varepsilon_{hk,ij} = u_{h,kij} + u_{k,hij} \quad (2.72)$$
$$-2 \varepsilon_{ih,jk} = -u_{i,hjk} + u_{h,ijk} \quad (2.73)$$
$$-2 \varepsilon_{jk,ih} = -u_{j,kih} + u_{k,jih} \quad (2.74)$$

Summing equations (2.71) to (2.74) yields the necessary condition to ensure the existence of the field $\bar{u}$.

$$\varepsilon_{ij,hk} + \varepsilon_{hk,ij} - \varepsilon_{ih,jk} - \varepsilon_{jk,ih} = 0 \quad (2.75)$$

Infinitesimal stretching ratio

When we work in the frame of linear deformations, i.e. with the infinitesimal strain tensor, the stretching ratio is given by the first order approximation of the ratio in (2.49), namely

$$\delta_i = \sqrt{1 + 2\varepsilon_{ii} - 1} \approx 1 + \frac{2\varepsilon_{ii}}{2} - 1 = \varepsilon_{ii} \quad (2.76)$$
Infiniteesimal angular dilatation

We invoke again the first order approximation of expression (2.57), so that, by replacing the finite strain tensor with the infiniteesimal strain tensor and by using the latter result for the stretching ratio, the angular dilatation assumes the following expression

$$\omega_{12} \simeq \frac{2\varepsilon_{12}}{\sqrt{1 + 2\varepsilon_{11}} \sqrt{1 + 2\varepsilon_{22}}} \simeq 2\varepsilon_{12} \quad (2.77)$$

With the proper subscripts shifting we can also write the angular dilatations $\omega_{23}$ and $\omega_{31}$.

Infiniteesimal area dilatation

Recalling equation (2.60), that is

$$\alpha = (1 + \delta_1) (1 + \delta_2) \cos \varphi'_{12}$$

the infiniteesimal area dilatation is obtained, as usual, by neglecting the second order terms, so that

$$\alpha \simeq \delta_1 + \delta_2 = \varepsilon_{11} + \varepsilon_{22} \quad (2.78)$$

Infiniteesimal volume dilatation

From equation (2.69), the first order approximation leads to the following expression for the infiniteesimal volumetric dilatation ratio

$$\nu \simeq \delta_1 + \delta_2 + \delta_3 \simeq \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{ij} \delta^{ij} \quad (2.79)$$

2.3.3 Deformation and rigid body motion

It is rather intuitive to understand that the motion of a flexible body can be made up of rigid translations and rotations as well as deformations. To see that from a mathematical point of view, consider the displacement field $\tilde{u}$ in a point $p$, as defined in (2.34), being defined by the first order approximation from the displacement $\tilde{u}_0$ on $p_0$.

$$u_j = u_{0j} + u_{j,i} dx_i$$

where it is clear that the translational component of the motion is wholly yielded by $u_{0j}$. Consequently the remaining part must store the deformation and rigid rotation components.
By observing that the gradient of $\bar{u}$ may be also written as follows

$$u_{j,i} = \frac{1}{2} (u_{j,i} + u_{i,j}) + \frac{1}{2} (u_{j,i} - u_{i,j})$$  \hspace{1cm} (2.80)

the displacement field becomes

$$u_j = u_{0j} + \frac{1}{2} (u_{j,i} + u_{i,j}) dx_i + \frac{1}{2} (u_{j,i} - u_{i,j}) dx_i =$$

$$= u_{0j} + \varepsilon_{ji} dx_i + \omega_{ji} dx_i$$  \hspace{1cm} (2.81)

where it has been put $\omega_{ji} = \frac{1}{2} (u_{j,i} - u_{i,j})$.

So, through the latter expression, the splitting of the displacement field $\bar{u}$ appears clear:

- $u_{0j}$: pure translation;
- $\varepsilon_{ij} = \frac{1}{2} (u_{j,i} + u_{i,j})$: pure deformation;
- $\omega_{ji} = \frac{1}{2} (u_{j,i} - u_{i,j})$: rigid body rotation.

In order to give a physical meaning to the operator curl introduced by equation (1.133) on page 26, we can observe that

$$\text{curl } \bar{u} = \epsilon_{kij} u_{j,i} \bar{e}_k =$$

$$= \frac{1}{2} \epsilon_{kij} (u_{j,i} + u_{i,j}) + \left( \frac{1}{2} \epsilon_{kij} (u_{j,i} - u_{i,j}) \right) \bar{e}_k =$$

$$= \epsilon_{kij} \omega_{ji} \bar{e}_k = \omega_k \bar{e}_k$$  \hspace{1cm} (2.82)

Using the identity (1.105) it is possible to prove$^4$ that the skew-symmetric component of a tensor is given by

$$\omega_{ji} = \frac{1}{2} \epsilon_{kij} \omega_k$$  \hspace{1cm} (2.83)

thus the rigid rotation turns into

$$\omega_{ji} dx_i = \frac{1}{2} \epsilon_{kij} \omega_k dx_i = \frac{1}{2} \bar{\omega} \times d\bar{l}$$  \hspace{1cm} (2.84)

where $\bar{\omega} = \text{curl } \bar{u}$ and $d\bar{l} = dx_i \bar{e}_i$.

$^4 \epsilon_{klp} \omega_k = \epsilon_{klp} \epsilon_{kij} \omega_{ji} = (\delta_{li} \delta_{pj} - \delta_{lj} \delta_{pi}) \omega_{ji} = 2 \omega_{pi}.$
2.4 Shell continuum

We define a shell–shaped region modeled on a surface $Q$ and with thickness $2\epsilon$ as a continuous medium $G(\epsilon)$ embedded in the Euclidean space $E$ each point of which is determined through a coordinate system $\{x^\alpha, \xi\} : G(\epsilon) \to \mathbb{R}^3$. Therefore, given $p^* \in G(\epsilon)$ it is defined by its position $p$ normally projected on $Q$ - by using the surface coordinate system introduced in (1.153) - and by the normal coordinate $\xi$ taken along the unit normal vector $\bar{n}$. In fact we have

$$p^* \mapsto (x^\alpha(p), \xi(p)) \quad (2.85)$$

The basis induced by the coordinate system $\{x^\alpha, \xi\}$ is $\{\partial_\alpha, \bar{n}\}$.

It is worthwhile pointing out that mechanics of shells - by virtue of such above statements - is traced back to the theory of surfaces, in fact vectors and tensors fields belonging to $T_\bar{Q}E$ will always be split into the parallel and normal components.

Note also that the symbol $\star$ denotes quantities belonging to the shell continuum.

2.4.1 General assumptions

The shell theory here introduced is based on the following hypotheses

**Hypothesis 1** The shell is sufficiently thin, so that

$$\frac{2\epsilon}{L} \ll 1 \quad L = \min \{R_{\text{min}}, L_{\text{min}}\} \quad (2.86)$$

where $R_{\text{min}}$ and $L_{\text{min}}$ are the minimum radius and a typical dimension of the shell structure, respectively.

**Hypothesis 2** (Linear theory) Displacements are infinitesimally small such that their products can be neglected in the kinematic expressions. This assumption allows us to write the equilibrium equations in the unstrained shell configuration.

**Hypothesis 3** The material filaments along the coordinate $\xi$ remain straight throughout the deformation and no length change is allowed. Namely, the distance between $p^* \in G(\epsilon)$ and the surface $Q$ is unaltered

$$\xi = \text{const.} \quad (2.87)$$
**Hypothesis 4 (Kirchhoff–Love theory)** The line elements initially normal to the shell’s mid-surface remain normal to it during the deformation.

\[ \bar{g}(\bar{\partial}_d, \bar{n}_d) = 0 \]  

(2.88)

where the subscript \( d \) is denotes quantities related to deformed configuration.

Note that the last hypothesis is nothing but the extension to a two–dimensional model of the Bernoulli theory for beams.

### 2.4.2 Strain tensor

A generic point \( p^* \in G(\epsilon) \) is determined by the vector \( r^* \) referred to the global Cartesian axes, so that

\[ r^* = \bar{r} + \xi \bar{n} \]  

(2.89)

where \( \xi \in (-\epsilon, \epsilon) \). See figure 2.5.

Let us suppose now that a quasi–static motion produces a deformed shell configuration points of which are univocally determined by the vector

\[ \bar{r}^*_d = \bar{r}_d + \xi_d \bar{n}_d \]  

(2.90)

where \( \xi_d \in (-\epsilon, \epsilon) \).

The displacement field is obtained by subtracting equations (2.89) and (2.90), so that

\[ \bar{r}^*_d - r^* = \bar{r}_d - \bar{r} + \xi (\bar{n}_d - \bar{n}) \]  

(2.91)

where we have made use of hypothesis 3. Equation (2.91) allows us to define the positional field as a function of two vector fields

\[ \bar{v} = \bar{r}_d - \bar{r} \quad v \in \bar{T}_Q E \]  

(2.92)

\[ \bar{w} = \bar{n}_d - \bar{n} \quad w \in \bar{T}Q \]  

(2.93)

To obtain the strain tensor no more theoretical concepts are required. We already know the definition and we just need to compute the metric tensors associated to the coordinate systems in the strained and the original configurations, so we have

\[ \gamma_{ij} = \begin{pmatrix} \gamma_{\alpha\beta} & \gamma_{\alpha3} \\ \gamma_{3\alpha} & \gamma_{33} \end{pmatrix} \]
Figure 2.5: Two dimensional sketch of the displacement field for Kirchhoff-Love shells.

where

\[ \gamma_{\alpha\beta} = \frac{1}{2} \left( g_{\alpha\beta d}^* - g_{\alpha\beta}^* \right) \]  \hspace{1cm} (2.94)

\[ \gamma_{\alpha 3} = \gamma_{3\alpha} = \frac{1}{2} \left( g_{\alpha 3d}^* - g_{\alpha 3}^* \right) \]  \hspace{1cm} (2.95)

\[ \gamma_{33} = \frac{1}{2} \left( \bar{n}_d \cdot \bar{n}_d - \bar{n} \cdot \bar{n} \right) = 0 \]  \hspace{1cm} (2.96)

According to equation (1.79) we have

\[ g_{\alpha\beta d}^* = \bar{\partial}_{\alpha d}^* \cdot \bar{\partial}_{\beta d}^* \]  \hspace{1cm} (2.97)

and

\[ g_{\alpha\beta}^* = \bar{\partial}_{\alpha}^* \cdot \bar{\partial}_{\beta}^* \]  \hspace{1cm} (2.98)

where, recalling equation (1.76), we can write

\[ \gamma_{\alpha\beta} = \frac{1}{2} \left[ \bar{\partial}_{\alpha d}^* \cdot \bar{\partial}_{\beta d}^* - \bar{\partial}_{\alpha}^* \cdot \bar{\partial}_{\beta}^* \right] = \]

\[ = \frac{1}{2} \left[ \left( \bar{\partial}_{\alpha d} + \xi \nabla_{\alpha} \bar{n}_d \right) \cdot \left( \bar{\partial}_{\beta d} + \xi \nabla_{\beta} \bar{n}_d \right) \right] + \]

\[ - \frac{1}{2} \left[ \left( \bar{\partial}_{\alpha} + \xi \nabla_{\alpha} \bar{n} \right) \cdot \left( \bar{\partial}_{\beta} + \xi \nabla_{\beta} \bar{n} \right) \right] = \]

\[ = \frac{1}{2} \left[ \bar{\partial}_{\alpha d} \cdot \bar{\partial}_{\beta d} + \bar{\partial}_{\alpha d} \cdot \xi \nabla_{\beta} \bar{n}_d + \bar{\partial}_{\beta d} \cdot \xi \nabla_{\alpha} \bar{n}_d + \xi^2 \nabla_{\alpha} \bar{n}_d \cdot \nabla_{\beta} \bar{n}_d \right] + \]

\[ - \frac{1}{2} \left[ \bar{\partial}_{\alpha} \cdot \bar{\partial}_{\beta} + \bar{\partial}_{\alpha} \cdot \xi \nabla_{\beta} \bar{n} + \bar{\partial}_{\beta} \cdot \xi \nabla_{\alpha} \bar{n} + \xi^2 \nabla_{\alpha} \bar{n} \cdot \nabla_{\beta} \bar{n} \right] \]  \hspace{1cm} (2.99)
where we realize that the tensor $\gamma_{\alpha\beta}$ can be split in three parts as follows

$$\gamma_{\alpha\beta} = \alpha_{\alpha\beta} + \xi \omega_{\alpha\beta} + \xi^2 \varphi_{\alpha\beta} \quad (2.100)$$

We define the stretching strain tensor as

$$\alpha_{\alpha\beta} = \frac{1}{2} \left[ \tilde{\alpha}_{\alpha\beta} - \tilde{g}_{\alpha\beta} \right] = \frac{1}{2} \left( g_{\alpha\beta} - g_{\alpha\beta} \right) \quad (2.101)$$

next, the first bending strain tensor as

$$\omega_{\alpha\beta} = \frac{1}{2} \left[ \tilde{\alpha}_{\alpha\beta} \cdot \nabla \tilde{\beta} \tilde{n}_d + \tilde{\beta}_{\alpha\beta} \cdot \nabla \tilde{\alpha} \tilde{n}_d - \tilde{\alpha}_{\alpha\beta} \cdot \nabla \tilde{\beta} \tilde{n} - \tilde{\beta}_{\alpha\beta} \cdot \nabla \tilde{\alpha} \tilde{n} \right] \quad (2.102)$$

and the second bending strain tensor as

$$\varphi_{\alpha\beta} = \frac{1}{2} \left[ \nabla \tilde{\alpha} \tilde{n}_d \cdot \nabla \tilde{\beta} \tilde{n}_d - \nabla \tilde{\alpha} \tilde{n} \cdot \nabla \tilde{\beta} \tilde{n} \right] \quad (2.103)$$

Considering now that the displacements are small enough to be negligible the second order terms

$$\nabla \tilde{\alpha} \tilde{v} \cdot \nabla \tilde{\beta} \tilde{v} \simeq 0$$
$$\nabla \tilde{\alpha} \tilde{v} \cdot \nabla \tilde{\beta} \tilde{w} \simeq 0$$

and recalling equations (2.92) and (2.93), the stretching and the bending strain tensors become, respectively

$$\alpha_{\alpha\beta} = \frac{1}{2} \left( \tilde{\alpha}_{\alpha\beta} \cdot \nabla \tilde{\beta} \tilde{v} + \tilde{\beta}_{\alpha\beta} \cdot \nabla \tilde{\alpha} \tilde{v} \right) = \frac{1}{2} \left( v_{\alpha|\beta} + v_{\beta|\alpha} + 2v^\xi L_{\alpha\beta} \right) \quad (2.104)$$

$$\omega_{\alpha\beta} = \frac{1}{2} \left( \tilde{\alpha}_{\alpha\beta} \cdot \nabla \tilde{\beta} \tilde{w} + \tilde{\beta}_{\alpha\beta} \cdot \nabla \tilde{\alpha} \tilde{w} \right) + \frac{1}{2} \left( \nabla \tilde{\alpha} \tilde{v} \cdot \nabla \tilde{\beta} \tilde{n} + \nabla \tilde{\beta} \tilde{v} \cdot \nabla \tilde{\alpha} \tilde{n} \right) = \frac{1}{2} \left( w_{\alpha|\beta} + w_{\beta|\alpha} + \gamma_{\alpha} L_{\gamma\beta} + v_{\beta} L_{\gamma\alpha} \right) + \frac{1}{2} \left( v^\xi \left( L^\gamma_{\alpha} L_{\gamma\beta} + L^\beta_{\gamma\alpha} \right) \right) \quad (2.105)$$

$$\varphi_{\alpha\beta} = \frac{1}{2} \left( w_{\alpha} \gamma_{\alpha\beta} + \gamma_{\beta} \gamma_{\alpha\gamma} \right) \quad (2.106)$$

where we have put

$$\nabla \tilde{\alpha} \tilde{v} = \left( v^\gamma_{\gamma\alpha} + v^\xi L^\gamma_{\alpha} \right) \tilde{\partial}_\gamma + \left( v^\xi_{\alpha} - v^\gamma \gamma_{\alpha\gamma} \right) \tilde{n} \quad (2.107)$$

$$v^\gamma_{\gamma\alpha} = v^\gamma_{\alpha} + \omega^\gamma \gamma_{\alpha\omega} \quad (2.108)$$
and

\[ \nabla_\alpha \bar{w} = w^\gamma_\alpha \bar{\partial}_\gamma - w^\gamma L_{\alpha\gamma} \bar{n} \]  
\[ w^\gamma_\alpha = w^\gamma_\alpha + w^\omega \Gamma^\gamma_{\alpha\omega} \]  

and

\[ \nabla_\alpha \bar{n} \cdot \nabla_\beta \bar{w} = L^\gamma_\alpha \bar{\partial}_\gamma \cdot \left( w^\omega_\beta \bar{\partial}_\omega - w^\omega L_{\beta\omega} \bar{n} \right) = L_{\omega\alpha} w^\omega_\beta \]  

Finally, the strain tensor assumes the following form

\[ \gamma_{\alpha\beta} = \frac{1}{2} \left( v_{\alpha|\beta} + v_{\beta|\alpha} + 2v^\xi L_{\alpha\beta} \right) + \frac{1}{2} \xi \left( w_{\alpha|\beta} + w_{\beta|\alpha} + v^\gamma_\alpha L_{\gamma\beta} + v^\gamma L_{\gamma\alpha} \right) + \frac{1}{2} \left( v^\xi \left( L^\gamma_\alpha L_{\gamma\beta} + L^\gamma_\beta L_{\gamma\alpha} \right) \right) + \frac{1}{2} \xi^2 \left( w^\gamma_\alpha L_{\gamma\beta} + w^\gamma_\beta L_{\gamma\alpha} \right) \]  

The stretching strain tensor does not depend on the thickness, in fact it describes the deformation of the mid--surface \( Q \). The bending strain tensors describe the deformation along the thickness.

The transversal components of the strain are

\[ \gamma_{3\alpha} = \gamma_{\alpha 3} = \frac{1}{2} \left( \bar{n}_d \cdot \bar{\partial}_{\alpha d} - \bar{n} \cdot \bar{\partial}_\alpha \right) = v^\xi_\alpha - v^\gamma L_{\alpha\gamma} + w_\alpha \]  

**Kirchhoff–Love strain theory**

If we take into account the *Kirchhoff-Love* hypothesis, see hypothesis 4, we have

\[ \bar{\partial}_{\alpha d} \cdot \bar{n}_d = 0 \Rightarrow (\bar{n} + \bar{w}) \cdot (\bar{\partial}_\alpha + \nabla_\alpha \bar{v}) = 0 \Rightarrow \]  
\[ \bar{w} \cdot \partial_\alpha + \bar{n} \cdot \nabla_\alpha \bar{v} = 0 \Rightarrow w_\alpha = v^\gamma L_{\alpha\gamma} - v^\xi_\alpha \]  

and we observe that the variables reduce just to the field \( \bar{v} \). Thus, the strain tensor turns into

\[ \alpha_{\alpha\beta} = \frac{1}{2} \left( v_{\alpha|\beta} + v_{\beta|\alpha} + 2v^\xi L_{\alpha\beta} \right) \]  
\[ \omega_{\alpha\beta} = v^\gamma_\alpha L_{\gamma\beta} + v^\gamma_\beta L_{\gamma\alpha} + v^\gamma L_{\gamma\alpha|\beta} - v^\xi_{\alpha\beta} + v^\xi L^\gamma_\alpha L_{\gamma\beta} \]
2\varphi_{\alpha\beta} = \xi^2 \left( v_\delta^\gamma L_\gamma L_\beta^\gamma + v_\delta^\gamma L_\gamma |_\alpha L_\beta^\gamma - v_\gamma^\xi L_\beta^\gamma \right) + \\
+ \xi^2 \left( v_\beta^\gamma L_\gamma L_\alpha^\gamma + v_\delta^\gamma L_\gamma |_\beta L_\alpha^\gamma - v_\gamma^\xi L_\alpha^\gamma \right) \tag{2.118}

In the linear theory the second bending strain tensor can be neglected because \( \xi \) is very small and its square makes the contribution of \( \varphi_{\alpha\beta} \) insignificant.

Finally, we have

\[ \gamma_{33} = \gamma_{3\alpha} = \gamma_{3\alpha} = 0 \tag{2.119} \]

Consider now a Cartesian coordinate system where all the Christoffel symbols vanish, we immediately realize the well known expression of the strain tensor for bending plates

\[ \gamma_{\alpha\beta} = \frac{1}{2} \left( v_{\alpha,\beta} + v_{\beta,\alpha} - 2v_{\alpha,\beta}^\xi \right) \tag{2.120} \]
Chapter 3
Analysis of stress

This chapter presents the classical stress analysis of a three-dimensional continuum subjected to both body and surface forces. It begins with the notions of stress vector and stress tensor which bring to enunciate the famous Cauchy's theorem, then the static equilibrium equations will be derived.

Next, the graphical representation through Mohr's circle and the principal directions associated with the state of stress will be also analyzed.

Curvilinear coordinate systems will be introduced only in the last section, where the analysis of stress for shell continuums will be shortly introduced.

3.1 Body and surface forces

Let \( V \) be the configuration of the continuous medium. We suppose that \( V \) is bounded by the closed surface \( S \). Consider a small region \( \Delta V \) subset of \( V \) and a small surface element \( \Delta S \) of \( S \). To analyze the forces acting on the volume element \( \Delta V \) it is necessary to account for two types of forces:

Body forces (or volume forces). These are the forces which are proportional to the mass contained in the volume element \( \Delta V \).

Surfaces forces. These are the forces being measured as force per unit area of surface \( \Delta S \) on which they act.

A good example of body forces is gravity: \( \rho g \Delta V \) - where \( \rho \) is the density of the continuum and \( g \) is the gravitational acceleration - or inertia.

Examples of surface forces are: pressure and tension \( \bar{t}_n (p, \bar{n}) \) (discussed in depth later on), which two parts of a continuum mutually exchange.
If we imagine that the continuous medium $\mathcal{V}$, equilibrium of which we are searching, is a subset of a bigger imaginary continuous body, then the tensions exchanged between these two portions can be assumed as external force loads.

In order to write the equations of equilibrium we consider both body forces $\vec{b} = b^i \vec{e}_i$ and surface forces $\vec{t}_n$. See figure 3.1.

The body forces also produce a resultant moment $\vec{M} = M^i \vec{e}_i$, where

$$\vec{M} = \int_{\mathcal{V}} (\vec{r} \times \vec{b}) \, d\mathcal{V} \quad (3.1)$$

### 3.2 State of stress

Let $\mathcal{V}$ be the configuration of a continuous medium, whose points are referred to a rectangular coordinate system

$$x^i : E \to \mathbb{R} : \quad p \mapsto \vec{g} \left((p - o), \vec{e}_i\right) \quad (3.2)$$

where $p$ and $o$ are points of $E$ and $\{\vec{e}_i\}$ is the unit normal basis of $E$.

Suppose on the body $\mathcal{V}$ surface and body forces, e.g. $\vec{b}$, $\vec{S}_j$, $\vec{M}_k$, $\vec{f}$ act in such a way to assure the equilibrium state. See figure 3.2. Due
to these forces throughout the body internal reactions are activated between the material points.

To understand the stress condition created at generic point $p$ within the body $\mathcal{V}$, we suppose to cut the continuous medium by means of a generic plane $\pi_n$, so that two portions $\mathcal{V}_1$ and $\mathcal{V}_2$ are produced.

![Figure 3.2: Body $\mathcal{V}$ being in an equilibrium state.](image)

After splitting, the portions of the body on the left and on the right side of the section plane $\pi_n$ lose their equilibrium state. In fact, before parting, both $\mathcal{V}_1$ and $\mathcal{V}_2$ were in equilibrium due to the mutual forces exchanged through the plane.

Cauchy enunciated the principle that, within a body, the forces that an enclosed volume imposes on the remainder of the body must be in equilibrium with the forces from the remainder of the body itself.

We denote by $\Delta A_n$ the small area surrounding $p$ and by $\Delta S_n$ and $\Delta M_n$ the force and the couple resultants in $p$ stemmed from the internal force distribution acting through $\Delta A_n$. See figure 3.3.

Cauchy’s principles implies the following limits

1. $\lim_{\Delta A_n \to 0} \frac{\Delta S_n}{\Delta A_n} = \bar{t}_n (p, \bar{n}) = -\bar{t}_n (p, -\bar{n})$

2. $\lim_{\Delta A_n \to 0} \frac{\Delta M_n}{\Delta A_n} = 0$
Figure 3.3: Splitting of the continuous media $\mathcal{V}$.

The vector $\bar{t}_n(p, \bar{n})$ is called the *Cauchy stress vector* and represents the surface force per unit of area acting at point $p$. The second limit assures that the entire state of stress for a fixed point $p$ is only defined by the forces, that is the couples are infinitesimal in comparison with them.

It’s important to observe that $\bar{t}_n$ is a linear mapping defined as follows

$$\bar{t}_n : E \times \bar{E} \rightarrow \bar{E}$$

so that

$$\bar{t}_n (p) \in L (\bar{E}, \bar{E})$$

and we recognize $\bar{t}_n (p)$ to be an endomorphism which is associated to a tensor belonging to $\bar{E}^* \otimes \bar{E}$.

In the following paragraphs this tensor will be thoroughly analyzed.
3.2.1 Stress vector components

Let $\bar{n}$ be the unit normal vector of the surface $\Delta A_n$ located at $p$. We can write the components of stress vector $\bar{t}_n(p, \bar{n})$ as follows

$$\bar{t}_n(p, \bar{n}) = t^i_n(p, \bar{n}) \bar{e}_i$$

(3.5)

so that the normal components of $\bar{t}_n(p)$ can be easily written as

$$t^n_n(p, \bar{n}) = \bar{n} \cdot \bar{t}_n(p, \bar{n}) = t^n_i(p, \bar{n}) n_i$$

(3.6)

Let us observe that the stress vector, which represents the entire state of stress at $p$, is completely known if the three coordinate components $t^i_n(p, \bar{n})$ are known.

3.2.2 Stress tensor

Now we want to show that the state of stress at any point of the continuum is entirely characterized specifying a linear mapping, i.e. endomorphism, represented by the nine quantities called components of stress tensor.

As usual, $p$ is a point in the medium and $\bar{t}_n(p, \bar{n})$ is the stress vector acting on the surface element passing for $p$ with the unit normal $\bar{n}$. Imagine to have four planar elements, three of which are parallel to the coordinate planes, the fourth one is supposed passing normal to $\bar{n}$, at a very small distance to $p$. We obtain a small tetrahedron. See figure 3.4

We shall denote by $\bar{t}_i$, with $i = 1, 2, 3$, the stresses vector$^2$ acting on the planar surface element orthogonal to the coordinate curves $x_i$, namely $\bar{t}_i = \bar{t}_i(p, \bar{e}_i)$. Evidently, every stress vector can be written by its components in the following way

$$\bar{t}_i = t^j_i \bar{e}_j \quad i, j = 1, 2, 3$$

(3.7)

where

$$t^j_i = \bar{e}_j \cdot \bar{t}_i$$

(3.8)

The forces acting on the tetrahedron are both surface and body forces.

---

$^1$It must be noted that in general the stress vector $\bar{t}_n(p, \bar{n})$ is not in the direction of $\bar{n}$.

$^2$Rigorously $\bar{t}_i$ should be write as $\bar{t}_{\bar{e}_i}(p)$. 
Figure 3.4: Stress vectors: the sketch of Cauchy’s theorem.

- Body forces: \( \bar{b}d\mathcal{V} \)

- Surface forces: \( -\bar{t}_i (p, -\bar{e}_i) \, d\mathcal{A}_i + \bar{t}_n (p, \bar{n}) \, d\mathcal{A}_n \), with \( i = 1, 2, 3 \).

thus the translational equilibrium of the tetrahedron can be readily written as

\[
- \bar{t}_i (p, -\bar{e}_i) \, d\mathcal{A}_i + \bar{t}_n (p, \bar{n}) \, d\mathcal{A}_n + \bar{b}d\mathcal{V} = 0 \quad (3.9)
\]

that taking into account that \( d\mathcal{A}_i = d\mathcal{A}_n n_i \), indeed we have \( n^i = \bar{n} \cdot \bar{e}_i = \cos (\bar{n}, \bar{e}_i) \), the above expression turns into

\[
-\bar{t}_i (p, -\bar{e}_i) \, d\mathcal{A}_n n_i + \bar{t}_n (p, \bar{n}) \, d\mathcal{A}_n + \frac{1}{3} \rho gh d\mathcal{A}_n = 0 \Rightarrow \quad (3.10)
\]

\[
-\bar{t}_i (p, -\bar{e}_i) n_i + \bar{t}_n (p, \bar{n}) + \frac{1}{3} \rho gh = 0 \quad (3.11)
\]

and, for \( h \) approaching zero, i.e. the infinitesimal volume surrounding \( p \), the equilibrium becomes

\[
\bar{t}_n (p, \bar{n}) = \bar{t}_i (p, -\bar{e}_i) n_i \quad (3.12)
\]
Figure 3.5: Stress tensor components.

Equation (3.12) represents Cauchy's theorem which states that the stress state \( \bar{t}_n (p, \bar{n}) \) can be completely determined by the stress vectors \( \bar{t}_i (p, -\bar{e}_i) \), acting on the face with outward unit normal vector \( -\bar{e}_i \), where \( \bar{n} \) is considered known. This result also proves that we are really dealing with a tensor as introduced by the endomorphism (3.4).

It will be convenient to use the customary notation, so that equation (3.12) may be rewritten in components as follows

\[
 t^j_n (p, \bar{n}) \bar{e}_j = t^j_i (p, -\bar{e}_i) n_i \bar{e}_j
\]

from which the stress tensor \( \sigma \) is defined as

\[
 t^j_n (p, \bar{n}) = \sigma_{ij} n_i
\]

The tensor \( \sigma_{ij} \) is called the stress tensor, it completely defines the state of stress at point \( p \) and repre-

---

\(^3\)Augustin Louis Cauchy (August 21, 1789 - May 23, 1857) was a French mathematician.

Source: http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Cauchy.html.
sents the component of the vector $\bar{t}_i$ working in direction of $x^j$.

See figure 3.5.
We can summarize saying that

$$\sigma : E \rightarrow L(\bar{E}, \bar{E}), \quad p \mapsto \sigma(p) \in \bar{E}^* \otimes \bar{E} \quad (3.15)$$

so

$$\sigma(p)(\bar{n}) = \bar{t}_n(p, \bar{n}) \quad (3.16)$$

that in components, (3.16), becomes

$$t^j_n = \sigma^j_h n^h \quad (3.17)$$

\textbf{Figure 3.6:} Stress tensor components acting on an infinitesimal volume element.

We remind the reader again that the lower and upper indices, in this context, can mutually be exchanged; moreover, they can be placed both upper and lower. So, generally, we shall also write

$$t_{nj} = \sigma_{nj} n^h \quad (3.18)$$

3.3 Equations of equilibrium

3.3.1 Translational equilibrium

With respect to the body $\mathcal{V}$, bounded by the closed surface $\mathcal{S}$, the condition of equilibrium requires that

$$\int_\mathcal{V} \bar{b} \, d\mathcal{V} + \int_\mathcal{S} \bar{t}_n \, d\mathcal{S} = 0 \quad (3.19)$$
Making use of (3.16), equation (3.19) becomes

$$\int_{V} \vec{b} \, dV + \int_{S} \sigma (p) (\vec{n}) \, dS = 0 \quad (3.20)$$

The divergence theorem can be applied to integral (3.20), so that

$$\int_{V} \vec{b} \, dV + \int_{V} \text{div} \sigma (p) \, dV = \int_{V} (\vec{b} + \text{div} \sigma (p)) \, dV = 0 \quad (3.21)$$

Since the region of integration $V$ is arbitrary, i.e. each part of the medium is in equilibrium, integral (3.21) vanishes, thus, at every point of $V$ we have

$$\text{div} \sigma + \vec{b} = 0 \quad (3.22)$$

that in components becomes

$$\sigma_{ij,i} + b_j = 0 \quad (3.23)$$

### 3.3.2 Rotational equilibrium

As well as the translational equilibrium, we require that the moments acting on the body are also in equilibrium, so

$$\int_{V} (\vec{r} \times \vec{b}) \, dV + \int_{S} (\vec{r} \times \vec{t}_n) \, dS = 0 \quad (3.24)$$

which in components, by virtue of the skew–symmetric tensor $\epsilon$, becomes

$$\int_{V} (r^i b^j \epsilon_{kij} \vec{e}_k) \, dV + \int_{S} (r^i \vec{e}_i \times \sigma_{jh} n_h \vec{e}_j) \, dS = 0 \Rightarrow$$

$$\int_{V} (r^i b^j \epsilon_{kij} \vec{e}_k) \, dV + \int_{S} (r^i \sigma_{jh} n_h \epsilon_{kij} \vec{e}_k) \, dS = 0$$

With the aid of the divergence theorem, for the $k$–th component we can write

$$\epsilon_{kij} \int_{V} (r^i b^j) \, dV + \epsilon_{kij} \int_{V} (r^i \sigma_{jh})_{,h} \, dV = 0 \Rightarrow$$

$$\epsilon_{kij} \int_{V} (r^i b^j + r^i_{,h} \sigma_{jh} + r_i \sigma_{jh,h}) \, dV = 0$$
Recalling equation (3.23), and that \( r_{i,h} = \delta_{ih} \) and since the volume \( V \) is arbitrary, the rotational equilibrium produces

\[
\epsilon_{kij} \sigma_{ij} = 0
\]  
\[(3.25)\]

Therefore equation (3.25) imposes the symmetry of the components of the stress tensor:

\[
\sigma_{ij} = \sigma_{ji}
\]  
\[(3.26)\]

The symmetry of the stress tensor can be also seen considering the volume element taken in shape of a rectangular parallelepiped, with faces parallel to the coordinate planes and with stress vector \( \bar{t}_i \) acting on the face perpendicular to the \( x_i \)-axis. Denoting the coordinates \( \{x_1, x_2, x_3\} \) with \( \{x, y, z\} \) - as often happens in literature - for the \((y, z)\)-plane the rotational equilibrium becomes

\[
(\sigma_{yz} \, dx \, dy) \, dz = (\sigma_{zy} \, dx \, dy) \, dz \Rightarrow \sigma_{yz} = \sigma_{zy}
\]  
\[(3.27)\]

See figure 3.7.

If we write the equilibrium for all planes, we obtain again the result in (3.26).

![Figure 3.7: Plane \((y, z)\). Components of the stress tensor acting on the volume element.](image)

### 3.3.3 Boundary equations

Let \( \hat{f} \) be the external force acting on the surface \( S_{\sigma} \) and \( \hat{u} \) the displacement field imposed on the remaining portion \( S_u \), so that
\( \mathcal{S} = \mathcal{S}_\sigma \cup \mathcal{S}_u \). Each point of \( \mathcal{V} \) lying on the boundary \( \mathcal{S} \) must satisfy the equilibrium and kinematic conditions as follows

\[
\tilde{t}_n dS = \tilde{f} dS, \quad \forall p \in \mathcal{S}_\sigma \tag{3.28}
\]

\[
\tilde{u} = \hat{u}, \quad \forall p \in \mathcal{S}_u \tag{3.29}
\]

that in components is

\[
\sigma_{nj} n_j = f_j, \quad \forall p \in \mathcal{S}_\sigma \tag{3.30}
\]

\[
u_i = \hat{u}_i, \quad \forall p \in \mathcal{S}_u \tag{3.31}
\]

### 3.4 Principal stresses and principal directions

Let us consider now the sheaf of planes passing through \( p \in \mathcal{V} \). Among the infinite planes there are some for which all the stress components vanish except the normal one. These planes are said \textit{principal planes} and their normal directions are said \textit{principal directions}. Hence, if \( \tilde{n} \) is a principal directions, by definition, we have at \( p \)

\[
\tilde{t}_n = \sigma \tilde{n}, \quad \sigma \in \mathbb{R} \tag{3.32}
\]

To find the three principal stresses we impose

\[
\tilde{t}_n = \sigma_{ih} n_h \tilde{e}_i = \sigma \tilde{n} \Rightarrow \sigma_{ih} n_h \tilde{e}_i = \sigma n_i \tilde{e}_i \tag{3.33}
\]

so that

\[
\sigma_{ih} n_h - \sigma \delta_{ih} n_h = 0 \Rightarrow (\sigma_{ih} - \sigma \delta_{ih}) n_h = 0 \tag{3.34}
\]

Expression (3.34) is a set of three homogeneous equations in the unknown direction \( \tilde{n} \). The solution is nonvanishing if, and only if, the determinant of the coefficients matrix is equal to zero; that is

\[
|\sigma_{ij} - \sigma \delta_{ij}| = 0 \tag{3.35}
\]

Solving the determinant above we obtain the cubic equation called \textit{secular equation} in the unknown stresses \( \sigma \)

\[
\sigma^3 - I_1 \sigma^2 - I_2 \sigma - I_3 = 0 \tag{3.36}
\]

The (3.35), (or (3.36)) has three real roots \( \sigma_I, \sigma_{II}, \sigma_{III} \), which are called \textit{principal stresses}. If \( \sigma \) in equation (3.34) is replaced by
any one of these eigenvalues, the resulting set of equations may
be solved for the corresponding direction $\bar{n}$. These directions, $\bar{n}_I$, $\bar{n}_{II}$, $\bar{n}_{III}$ are called principal directions. The planes normal to the
principal directions are termed principal planes of stresses. In other
words we say that along the principal planes of stresses there is no
shearing stress.

Generally there are only three mutually orthogonal principals
directions.

The three scalars in (3.36) are invariants as regards to the co-
ordinate system. They are

$$I_1 = \text{tr}\sigma$$
$$I_2 = \frac{1}{2} (\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij})$$
$$I_3 = \det (\sigma_{ij})$$

These invariants are physically very important, they in fact al-
low us to characterize the stress state as follows

if $I_3 = 0$ : triaxial state of stress
if $I_3 = 0$ and $I_2 \neq 0$ : biaxial state of stress
if $I_3 = I_2 = 0$ and $I_1 \neq 0$ : axial state of stress

Now we want to point out that the principal stresses found solv-
ing equation (3.35) represent the maximum and minimum stress. To
see this we make use of the Lagrange multipliers method in order
to find the extremes of a function of several variables subjected to
one or more constraints. In this case recalling formulae (1.33) and
(1.34) we can write the stress tensor in a generic unknown coor-
dinate system rotated with respect the initial system as follows

$$\sigma'_{ij} = a^i_h \sigma^h_k a'^k_j$$

(3.37)

and we also impose the constraint on the unknown matrices $a$ and
$a'$ such as they are effectively two orthogonal transformations, i.e.
the condition (1.21). Hence we have

$$\mathcal{L} (a^i_h, \lambda) = a^i_h \sigma^h_k a'^k_j - \lambda \left( a^i_h a'^h_j - \delta^i_j \right)$$

(3.38)
and the stationary conditions are

\[
\frac{\partial L}{\partial a_i^h} = \sigma_k^h a_j^h - \lambda a_j^h = \left( \sigma_k^h - \lambda \delta_k^h \right) a_j^h = 0 \quad (3.39)
\]

\[
\frac{\partial L}{\partial a_i^h} = a_i^h a_j^h - \delta_j^i = 0 \quad (3.40)
\]

The first equation, (3.39), yields the following condition

\[
|\sigma_k^h - \lambda \delta_k^h| = 0 \quad (3.41)
\]

that is exactly the condition (3.35), hence we can derive that given a generic state of stress \(\sigma_{ij}\), the principal stresses \(\sigma_I, \sigma_{II}, \sigma_{III}\) are extrem values.

### 3.4.1 Normal and tangential components of the stress vector

The last equations of the previous section enable us to know the components of the stress vector for every direction we want. Let \(\tilde{n}\) be the unit normal vector and \(\tilde{\nu}\) the unit tangent vector. It follows that the normal and tangent components of the stress vector, \(\sigma\) and \(\tau\), respectively, are readily computed through the usual scalar product as follows

\[
\sigma = \tilde{t}_n \cdot \tilde{n} = t_{jn} (p, \tilde{n}) n_j = \sigma_{ij} n_i n_j \quad (3.42)
\]

\[
\tau = \tilde{t}_n \cdot \tilde{\nu} = t_{jn} (p, \tilde{n}) \nu_j = \sigma_{ij} n_i \nu_j \quad (3.43)
\]
From figure 3.8 on the preceding page it is also clear that the square tangent component of the stress vector can be written as follows

$$\tau^2 = |\vec{t}_n|^2 - \sigma$$  \hspace{1cm} (3.44)

### 3.4.2 Mohr’s circles

#### Two dimensional state of stress

An important graphical interpretation of the above results is due to O. Mohr\(^4\). Following [10], [9] and [12] let us begin considering the above relations (3.42) and (3.43) for a two dimensional problem, so that

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$  \hspace{1cm} (3.45)

The unit vectors $\vec{n}$ and $\vec{\nu}$, with respect to figure 3.9, have the following components

$$\vec{n} = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} \quad \text{and} \quad \vec{\nu} = \begin{pmatrix} -\cos \left(\frac{\pi}{2} - \phi\right) \\ \sin \left(\frac{\pi}{2} - \phi\right) \end{pmatrix} = \begin{pmatrix} -\sin \phi \\ \cos \phi \end{pmatrix}$$

Hence, the normal and tangent components of $\vec{t}_n (p, \vec{n})$ are

$$\sigma = \sigma_{11} \cos^2 \phi + \sigma_{22} \sin^2 \phi - 2\sigma_{12} \sin \phi \cos \phi$$  \hspace{1cm} (3.46)

$$\tau = -\sigma_{11} \cos \phi \sin \phi + \sigma_{22} \sin \phi \cos \phi + \sigma_{12} \cos^2 \phi - \sigma_{21} \sin^2 \phi$$  \hspace{1cm} (3.47)

that through some trigonometric manipulations\(^5\) turn respectively

\(^4\)Christian Otto Mohr October 8, 1835 - October 2, 1918 was a German civil engineer.

\(^5\)In particular these two identities have been used:

i) $2 \sin \phi \cos \phi = \sin 2\phi$,

ii) $\cos^2 \phi - \sin^2 \phi = \cos 2\phi$.
Figure 3.9: Normal and tangential components of the stress vector in two dimensions.

\begin{align*}
\sigma &= \frac{1}{2} \sigma_{11} \cos^2 \varphi + \frac{1}{2} \sigma_{11} \left(1 - \sin^2 \varphi\right) + \\
&\quad + \frac{1}{2} \sigma_{22} \sin^2 \varphi + \frac{1}{2} \sigma_{22} \left(1 - \cos^2 \varphi\right) + \sigma_{12} \sin 2\varphi = \\
&\quad = \frac{1}{2} (\sigma_{11} + \sigma_{22}) + \frac{1}{2} (\sigma_{11} - \sigma_{22}) \cos 2\varphi + \sigma_{12} \sin 2\varphi (3.48) \\
\tau &= -\sigma_{11} \cos \varphi \sin \varphi + \sigma_{22} \sin \varphi \cos \varphi + \sigma_{12} \cos^2 \varphi - \sigma_{21} \sin^2 \varphi = \\
&\quad = -\frac{1}{2} (\sigma_{11} - \sigma_{22}) \sin 2\varphi + \sigma_{12} \cos 2\varphi (3.49)
\end{align*}

Next, by squaring both terms of the latter equations and summing term by term, the variable $2\varphi$ disappears, hence

\begin{align*}
\left(\sigma - \frac{1}{2} (\sigma_{11} + \sigma_{22})\right)^2 + \tau^2 &= \left(\frac{1}{2} (\sigma_{11} - \sigma_{22})\right)^2 + \sigma_{12}^2 (3.50)
\end{align*}

If we represent the above equation in a two dimensional Cartesian system with $\sigma$ and $\tau$ as abscissa and ordinate, respectively, we realize it represents the equation of a circle in the form $(x - x_C)^2 + (y - y_C)^2 = R^2$ where

\begin{align*}
x_C &= \frac{1}{2} (\sigma_{11} + \sigma_{22}) \quad (3.51) \\
y_C &= 0 \quad (3.52)
\end{align*}

are the coordinates of the center and

\begin{align*}
R &= \sqrt{\left(\frac{1}{2} (\sigma_{11} - \sigma_{22})\right)^2 + \sigma_{12}^2} \quad (3.53)
\end{align*}
is the radius of the circle. This circle is known as \textit{Mohr’s circle} and it represents all the possible states of stress in \( p \). Namely, there exists a one-to-one relationship between each state of stress \( \bar{t}_n (p, \bar{n}) \), i.e. \( \sigma \) and \( \tau \), and points belonging to the circle. To show that, let us assume \( \gamma \) is the angle between the \( x_1 \)-axis and the stress vector \( \bar{t}_n \), as figure 3.10 depicts. To find the correspondence between the stress state and the circle let us observe that \( \gamma \) defines a principal direction so, by definition \( \tau = 0 \), and we have

\[
\tan 2\gamma = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} \quad (3.54)
\]

and with the aid of picture 3.10 in the above equation we recognize that \( P_1P^* = 2\sigma_{12} \) and \( P_2P^* = \sigma_{11} - \sigma_{22} \). Consequently the following expressions hold

\[
\begin{align*}
   r \cos 2\gamma &= \frac{1}{2} (\sigma_{11} - \sigma_{22}) \quad (3.55) \\
   r \sin 2\gamma &= \sigma_{12} \quad (3.56) \\
   \end{align*}
\]

that substituted into (3.48) and (3.49) and making use of some trigonometric identities\textsuperscript{6} yield, respectively

\[
\begin{align*}
   \sigma &= \frac{1}{2} (\sigma_{11} + \sigma_{22}) + r \cos 2(\gamma - \varphi) \quad (3.58) \\
   \tau &= r \sin 2(\gamma - \varphi) \quad (3.59)
   \end{align*}
\]

Thus, given a generic plane oriented as \( \varphi \) equations (3.58) and (3.59) are a parametric representation of a circle and so a one-to-one relationship between the state of stress and the Mohr’s circle is established. See figure 3.10.

We define \( P^* \equiv (\sigma_{11}, -\sigma_{12}) \) as the \textit{pole} of the circle. The line passing through \( P^* \) having inclination \( \varphi \) with respect to the vertical

\textsuperscript{6}In particular these identities have been used:

\textit{i)} \quad \cos 2\gamma \cos 2\varphi = \frac{1}{2} (\cos 2 (\gamma + \varphi) + \cos 2 (\gamma - \varphi)),

\textit{ii)} \quad \sin 2\gamma \sin 2\varphi = \frac{1}{2} (\cos 2 (\gamma - \varphi) - \cos 2 (\gamma + \varphi)),

\textit{iii)} \quad \sin 2\varphi \cos 2\varphi = \frac{1}{2} (\sin 2 (\gamma + \varphi) + \sin 2 (\gamma - \varphi)).
line joining $P_1$ and $P^*$ intersects the circle in $P_\varphi$ and the angle $\overline{P_\varphi CS_1}$ is right $2(\varphi - \gamma)$, so the coordinates of the point $P_\varphi$, in the $(\sigma, \tau)$–plane, are just those expressed by equations (3.58) and (3.59).

Thus, we have proved that, given a stress vector orientated as $\gamma$, once Mohr’s circle is known, the normal and tangent components of a stress vector can be graphically found provided the inclination $\varphi$ in known.

On the other hand, if the normal and tangent stresses are known, Mohr’s circle enables us to find directly the principal direction. In fact point $S_1$ has coordinates $\sigma = OC + R$ and $\tau = 0$, so that the line $P^*S_1$ defines the angle $\gamma$ that fixes the principal direction. See figure 3.10.

\begin{figure}[h]
\includegraphics[width=\textwidth]{figure310.png}
\caption{Normal and tangential components of the stress vector for in two dimensions.}
\end{figure}

Two other relevant features on Mohr’s circle are those for which the tangent component of $\bar{t}_n(p, \bar{n})$ is maximum. These directions can be found through the same procedure. Indeed the lines $P^*T_1$ and $P^*T_2$ represent the directions along which the stress vector has maximum shear component. See figure 3.10 and 3.11. Analytically these maximum and minimum values are

\begin{align}
\tau_{\text{max}} &= \frac{1}{2} \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2} \\
\tau_{\text{min}} &= -\frac{1}{2} \sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2}
\end{align}
and the directions can be computed putting zero the following derivative

\[
\frac{d\tau}{d\gamma} = 0 \Rightarrow \cos 2(\gamma - \varphi) = \frac{\pi}{2} \Rightarrow \gamma = \varphi + \frac{\pi}{4}
\]  

(3.62)

Let us summarize now the key items to draw and use the Mohr’s circle when a plane state of stress \(\sigma_{ij}\), with \(i, j = 1, 2\) is known with respect to a generic system \(\{x_1, x_2\}\). See figure 3.11.

1. Compute the radius \(R\) and the abscissa of the center \(C\) of the circle, equations (3.51) and (3.53);

2. Identify the pole \(P^*\);

3. Identify the principal direction drawing a line from \(P^*\) to both \(S_1\) and \(S_2\). The inclination of the latter defines the principal directions;

4. Compute the principal stresses \(\sigma_I\) and \(\sigma_{II}\) at the extreme points \(S_1\) and \(S_2\), respectively;

5. Compute the maximum and minimum shear stresses \(\tau_{\text{min}}\) and \(\tau_{\text{max}}\).

\[\text{Figure 3.11: Graphical determination of principal directions.}\]
Three dimensional state of stress

Consider again the state of stress in \( p \) referenced to the principal axes and let the principal stresses be ordered according to \( \sigma_I > \sigma_{II} > \sigma_{III} \). Assume the three principal stresses are known, so that, in accordance with equations (3.42) and (3.44), we write

\[
\sigma = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_{III} n_3^2
\]

\[
\tau^2 + \sigma^2 = (\sigma_I n_1)^2 + (\sigma_{II} n_2)^2 + (\sigma_{III} n_3)^2
\]

and being \( \bar{n} \cdot \bar{n} = n_1^2 + n_2^2 + n_3^2 = 1 \), by solving for the directions \( n_i \), we obtain

\[
\begin{align*}
n_1^2 &= \frac{\tau^2 + (\sigma - \sigma_{II})(\sigma - \sigma_{III})}{(\sigma_I - \sigma_{II})(\sigma_I - \sigma_{III})} \\
n_2^2 &= \frac{\tau^2 + (\sigma - \sigma_{III})(\sigma - \sigma_I)}{(\sigma_{II} - \sigma_{III})(\sigma_{II} - \sigma_I)} \\
n_3^2 &= \frac{\tau^2 + (\sigma - \sigma_I)(\sigma - \sigma_{II})}{(\sigma_{III} - \sigma_I)(\sigma_{III} - \sigma_{II})}
\end{align*}
\]

In the above equations \( \sigma_I, \sigma_{II}, \sigma_{III} \) are known; \( \sigma \) and \( \tau \) are functions of \( n_i \).

In order to interpret these equations graphically we note that in equation (3.63) \( \sigma_I - \sigma_{II} > 0 \) and \( \sigma_I - \sigma_{III} > 0 \), and \( n_1^2 \) is positive. Therefore

\[
(\sigma - \sigma_{II})(\sigma - \sigma_{III}) + \tau^2 \geq 0
\]

When the equality sign holds, this equation may be rewritten as

\[
\left[ \sigma - \frac{1}{2}(\sigma_{II} + \sigma_{III}) \right]^2 + \tau^2 = \left[ \frac{1}{2}(\sigma_{II} - \sigma_{III}) \right]^2
\]

which is the equation of a circle in the \((\sigma, \tau)\)-plane, where we assume \( \sigma \) as abscissa and \( \tau \) as ordinate. The circle in figure 3.12 is termed \( C_1 \) and has the center in \( \frac{1}{2}(\sigma_{II} + \sigma_{III}) \) on the \( \sigma \) axis, and radius \( \frac{1}{2}(\sigma_{II} - \sigma_{III}) \).

Examining equation (3.64) we observe that \( \sigma_{II} - \sigma_{III} > 0 \) and \( \sigma_{II} - \sigma_I < 0 \), so we have

\[
(\sigma - \sigma_{III})(\sigma - \sigma_I) + \tau^2 \leq 0
\]
The boundary of the area of this equation, i.e. in the case of equality sign, defines a circle as before in the \((\sigma, \tau)\)-plane

\[
\left[ \sigma - \frac{1}{2} (\sigma_I + \sigma_{III}) \right]^2 + \tau^2 = \left[ \frac{1}{2} (\sigma_I - \sigma_{III}) \right]^2
\]  

(3.69)

named \(C_2\). See figure 3.12.

The same procedure allows us to obtain from equation (3.65) the circle \(C_3\), indeed, we have the following condition

\[(\sigma - \sigma_I) (\sigma - \sigma_{III}) + \tau^2 \geq 0\]  

(3.70)

that at the boundary yields

\[
\left[ \sigma - \frac{1}{2} (\sigma_I + \sigma_{III}) \right]^2 + \tau^2 = \left[ \frac{1}{2} (\sigma_I - \sigma_{III}) \right]^2
\]  

(3.71)

Finally, from inequalities (3.66), (3.68), (3.70), it follows that admissible values of \(\sigma\) and \(\tau\) lie in the shaded region of figure 3.12 bounded by the circles \(C_1, C_2, C_3\). The value \(\tau_{\text{max}}\) and \(\sigma_{\text{max}}\) can be readily provided from figure 3.12, so that

\[
\tau_{\text{max}} = \frac{1}{2} (\sigma_I - \sigma_{III})
\]  

(3.72)

\[
\sigma_{\text{max}} = \frac{1}{2} (\sigma_I + \sigma_{III})
\]  

(3.73)

and as a consequence, the surface elements supporting these stresses are found replacing the above values into equations (3.63) to (3.65). For further details the reader is referred to [1].
3.5 Stress quadric of Cauchy

Consider an element of area $dA$ with a normal vector $\bar{n}$. As previously stated, the stress vector $\bar{t}_n(p, \bar{n})$ can be decomposed into a normal component $\sigma$ and a tangential component $\tau$.

Let us introduce now a local system of axes $\xi_i$ with origin in $p$ equipped with the unit normal basis $\{\bar{e}_i\}$. See figure 3.13.

![Figure 3.13: Stress quadratic of Cauchy.](image)

Let $\bar{r}$ be the vector taken along $\bar{n}$ joining $p$ with a generic point $p'$, namely $(p' - p) = r\bar{n}$. This vector can be equivalently expressed by the following expressions

$$\bar{r} = r\bar{n} \quad (3.74)$$
$$\bar{r} = \xi_i \bar{e}_i \quad (3.75)$$

The first equation provides the $j$-th component of $\bar{r}$ as follows

$$\xi_i = \bar{r} \cdot \bar{e}_i = r n_i \quad (3.76)$$

that, replaced into the expression of the normal component of the stress vector, yields the following relation

$$r^2 \sigma = \sigma_{ij} \xi_i \xi_j \quad (3.77)$$

We recognize equation (3.77) as a quadric form.$^7$

---

$^7$We remind that any quadric form $F$ can be expressed as

$$F(\bar{v}) = M_{ij} v_i v_j \quad (3.78)$$

where $\bar{v} = (v_1, v_2, v_3)^T$ is a vector expressed with respect to the chosen basis, and $M_{ij}$ is a certain symmetric matrix that depends on $F$ and on the basis.
So we restrict the coordinates of \( \xi_i \) by requiring the end point \( p' \) of \( \bar{r} \) to lie on the quadric surface

\[
F(\xi_1, \xi_2, \xi_3) = \sigma_{ij} \xi_i \xi_j = \pm k^2
\]  
(3.79)

where \( k \) is an arbitrary real constant and where the sign is chosen in such a way to make the surface real. As a result we have

\[
\sigma = \pm \frac{k^2}{r^2}
\]  
(3.80)

Since \( r^2 \) is a positive quantity, \( k^2 \) will be taken with the positive sign whenever the normal component \( \sigma \) is a tension and with negative sign when it represents compression.

Next, deriving equation (3.79) and by using equation (3.76), we obtain

\[
\frac{\partial F}{\partial \xi_i} = \sigma_{ij} \xi_j = \sigma_{ij} r n_j = r t^i_n (p, \bar{n})
\]  
(3.81)

which allows us to realize that the quadric form (3.79) has some properties of a potential function, indeed the partial derivative of \( F \) with respect the \( i \)-th coordinate gives, except for the constant \( r \), the force component (i.e. the component of the stress vector) right in the \( i \)-th direction.

Furthermore we observe that the above derivatives, equation (3.81), denote the direction of the normal \( \bar{n} \) to the plane tangent to the quadric surface (3.79) at point \( p' \), so that the right-hand term of equation (3.81) just establishes the stress vector \( \bar{t}_n (p, \bar{n}) \) is also normal to this tangent plane.

The above results have been directly taken from [1], to which the reader is referred for any further detail.

### 3.6 Stress–deviator and spherical components of the stress tensor

Every state of stress \( \sigma_{ij} \) may be decomposed into a spherical portion and into a portion \( s_{ij} \) known as stress–deviator by the following equation

\[
\sigma_{ij} = \sigma_M \delta_{ij} + s_{ij}
\]  
(3.82)

where \( \sigma_M = \frac{1}{3} \sigma_{ii} \) is the arithmetic mean of the normal stress, i.e. spherical stress component (or hydrostatic stress). Equation (3.82)
may be solved for $s_{ij}$

$$s_{ij} = \sigma_{ij} - \sigma_M \delta_{ij}$$  \hspace{1cm} (3.83)

where the latter components are termed stress–deviations.

Namely,

$$s_{ij} = \begin{pmatrix} \sigma_{11} - \sigma_M & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_M & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_M \end{pmatrix}$$

It is possible to prove that both the stress tensor $\sigma$ and the deviator tensor $s$ have the same principal directions. The characteristic equation for the deviator is

$$s^3 + J_2 s + J_3 = 0$$  \hspace{1cm} (3.84)

where the deviator invariants are

$$J_2 = -\frac{1}{2} s_{ij} s_{ij}$$

$$J_3 = \det s_{ij}$$

### 3.7 Stress in shell continuums

#### 3.7.1 Shifters

Before reasoning upon the stress state characterizing a shell continuum it is worth introducing some geometrical relations linking points belonging to the mid–surface $Q$ with corresponding points belonging to the shell thought as a three–dimensional continuum.

Therefore, let us recall the relation already met to compute the components of the metric tensor $g^*_{\alpha\beta}$, see equation (2.99) on page 53, between the basis in $p^* \in G(\epsilon)$ and the basis in $p$ projection of $p^*$ on $Q$ along the normal coordinate curve $\xi$. So we have

$$\bar{\partial}^*_\alpha = \bar{\partial}_\alpha + \xi L^\beta_{\alpha} \bar{\partial}_\beta$$  \hspace{1cm} (3.85)

$$\bar{n} = \bar{n}^*$$  \hspace{1cm} (3.86)

which in a short notation assumes the following form

$$\bar{\partial}^*_i = S^h_i \bar{\partial}_h$$  \hspace{1cm} (3.87)
Hence, with respect to the basis associated to the coordinate system \( \{ x^\alpha, \xi \} \) the tensor \( S \) has the following components

\[
S^h_i = \begin{pmatrix}
1 + \xi L_1^1 & \xi L_1^2 & 0 \\
\xi L_2^1 & 1 + \xi L_2^2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Therefore, the superficial part of \( S \) can be expressed by the following tensor product

\[
S^\dagger = d^\gamma \otimes \bar{\partial}_{\gamma}^\star
\]

so that

\[
S^\dagger(\bar{\partial}_{\beta}) = (d^\gamma \otimes \bar{\partial}_{\gamma}^\star)(\bar{\partial}_{\beta}) = \bar{\partial}_{\beta}^\star
\]

In the same way we define \( F^\dagger \) as follows

\[
F^\dagger = \bar{\partial}_{\gamma} \otimes d^{\star\gamma}
\]

so that

\[
F^\dagger(d^{\beta}) = (\bar{\partial}_{\gamma} \otimes d^{\star\gamma})(d^{\beta}) = d^{\star\beta}
\]

Tensors \( S^\dagger \) and \( F^\dagger \) are called shifter tensors.

### 3.7.2 Contraction of surface forces

Consider now a curve \( c : IR \to Q \) representing the intersection of the surface \( Q_c \) normal to \( Q \) which splits the shell continuum \( G(\epsilon) \) into two portions.

Let \( \bar{\nu} \in TQ \) be the unit normal vector applied in \( p \) outward pointing from \( c \) and let \( \bar{l} \in TQ \) be the unit vector tangent to \( c \) applied in the same point. Then the three unit vectors \( \{ \bar{\nu}, \bar{l}, \bar{n} \} \) form a local basis in \( p \). A similar triplet of vectors can be defined in \( p^\star \) as \( \{ \bar{\nu}^\star, \bar{l}^\star, \bar{n} \} \). Note that the symbol \( \star \) denotes as usual quantities belonging to the shell thickness. See figure 3.14.

In order to ensure the equilibrium condition, the portion of the shell included by \( Q_c \) must exert on the remaining part of the continuum a tension such as for each point \( p^\star \) is entirely described by the stress vector \( \bar{l}^\star \). Moreover the stress vector \( t^\star \) can be equivalently expressed by Cauchy stress tensor as follows

\[
\bar{l}^\star(p^\star, \bar{\nu}^\star) = \sigma^\star(p^\star) \bar{\nu}^\star
\]
where $\sigma^*$ is the contravariant form of the stress tensor defined in $p^*$. For the sake of brevity hereafter $\sigma^*(p^*)$ will be denoted simply by $\sigma$.

Now our goal is to establish a relation between the stress state distributed along the surface $Q_c$ and the stress state along the boundary of the mid-surface of the shell. This can be done by means of a reduction, i.e. a contraction, of the stress per unit area to a stress per unit line.

Therefore, let us define two vector fields $\mathbf{n}$ and $\mathbf{m}$ such as

$$\int_c \mathbf{n}(p, \nu)dl = \int_{Q_c} \bar{\ell}^*(p^*, \nu^*)dA^* \quad (3.93)$$

$$\int_c \mathbf{m}(p, \nu)dl = \int_{Q_c} ((p^* - p) \times \bar{\ell}^*(p^*, \nu^*))dA^* \quad (3.94)$$

Equalities (3.93) and (3.94) guarantee that the stress system $\mathbf{n}$ and $\mathbf{m}$ is statically equivalent to the stress system $\bar{\ell}^*$ along the fiber $\xi$ passing through $p$.

The oriented elemental area in equations (3.93) and (3.94) with respect to the local basis $\{\bar{\nu}^*, \bar{\ell}^*, \bar{n}\}$ is given by the following vectorial product

$$\nu^*dA^* = d\ell^* \times d\xi \bar{n} \quad (3.95)$$
and since $dl^* = dl^\alpha \bar{\partial}_\alpha$, equation (3.95) can be equivalently expressed as follows

$$\nu dl^* A^* = dl^\alpha \bar{\partial}_\alpha \times d\xi \bar{n} = \eta_{\alpha\beta} dl^\alpha d\xi \bar{d}^{*\beta} = \epsilon_{\alpha\beta} \sqrt{g^*} dl^\alpha d\xi \bar{d}^{*\beta}$$ (3.96)

where $g^* = \det (g^*_{\alpha\beta})$.

Moreover, back to the mid-surface we notice it is possible to write

$$dl^\alpha \bar{\partial}_\alpha \times \bar{n} = \nu dl$$ (3.97)

which in the coordinate system $\{x^\alpha, \xi\}$ becomes

$$dl^\alpha \bar{\partial}_\alpha \times \bar{n} = \eta_{\alpha\beta} dl^\alpha \bar{d}^{*\beta} = \epsilon_{\alpha\beta} \sqrt{g} dl^\alpha \bar{d}^{*\beta}$$ (3.98)

where $g = \det (g_{\alpha\beta})$.

Equation (3.92) and (3.96) allow us to rewrite equations (3.93) and (3.94) as follows

$$\int_c n(p, \nu) dl = \int_{Q_c} \sigma \epsilon_{\alpha\beta} \sqrt{g^*} dl^\alpha d\xi \bar{d}^{*\beta}$$ (3.99)

$$\int_c m(p, \nu) dl = \int_{Q_c} (p^* - p) \times \sigma \epsilon_{\alpha\beta} \sqrt{g^*} dl^\alpha d\xi \bar{d}^{*\beta}$$ (3.100)

Next, by virtue of the shifter $F^\dagger$, the latter equations become

$$\int_c n(p, \nu) dl = \int_{Q_c} \sigma \epsilon_{\alpha\beta} \sqrt{g^*} dl^\alpha d\xi \left( \bar{\partial}_\gamma \otimes \bar{d}^{*\gamma} \right) \bar{d}^{\beta}$$ (3.101)

$$\int_c m(p, \nu) dl = \int_{Q_c} \xi \bar{n} \times \sigma \epsilon_{\alpha\beta} \sqrt{g^*} dl^\alpha d\xi \left( \bar{\partial}_\gamma \otimes \bar{d}^{*\gamma} \right) \bar{d}^{\beta}$$ (3.102)

which, taking into account equations (3.97) and (3.98), become

$$\int_c n(p, \nu) dl = \int_c \int_{-\epsilon}^{+\epsilon} \sqrt{\frac{g^*}{g}} \sigma \left( \bar{\partial}_\gamma \otimes \bar{d}^{*\gamma} \right) \nu dl d\xi$$ (3.103)

$$\int_c m(p, \nu) dl = \int_c \int_{-\epsilon}^{+\epsilon} \xi \bar{n} \times \sqrt{\frac{g^*}{g}} \sigma \left( \bar{\partial}_\gamma \otimes \bar{d}^{*\gamma} \right) \nu dl d\xi$$ (3.104)

and finally

$$n(p, \nu) = \int_{-\epsilon}^{+\epsilon} g \sigma \left( \bar{\partial}_\gamma \otimes \bar{d}^{*\gamma} \right) \nu d\xi$$ (3.105)

$$m(p, \nu) = \bar{n} \times \int_{-\epsilon}^{+\epsilon} \xi g \sigma \left( \bar{\partial}_\gamma \otimes \bar{d}^{*\gamma} \right) \nu d\xi$$ (3.106)
where we have put $g = \sqrt{g^* g}$.

Both integrands in (3.105) and (3.106) can be further simplified just substituting $\sigma = \sigma^{ij} (\bar{\partial}_i^* \otimes \bar{\partial}_j^*)$ and $\nu = \nu_\alpha d^\alpha$ as follows

\[
\mathbf{n}(p, \nu) = \left( \int_{-\epsilon}^{+\epsilon} g \sigma^{\alpha j} \bar{\partial}_j^* d\xi \right) \nu_\alpha \quad (3.107)
\]
\[
\mathbf{m}(p, \nu) = \bar{n} \times \left( \int_{-\epsilon}^{+\epsilon} g \xi \sigma^{\alpha j} \bar{\partial}_j^* d\xi \right) \nu_\alpha \quad (3.108)
\]

and using once again equations (3.85) and (3.86) they assume the following form

\[
\mathbf{n}(p, \nu) = \left( \int_{-\epsilon}^{+\epsilon} g \sigma^{\alpha \gamma} d\xi + \int_{-\epsilon}^{+\epsilon} g \sigma^{\alpha \beta} \xi d\xi L_\beta^\gamma \right) \bar{\partial}_\gamma \nu_\alpha + \\
\left( \int_{-\epsilon}^{+\epsilon} g \sigma^{\alpha \xi} d\xi \right) \bar{n} \nu_\alpha \quad (3.109)
\]
\[
\mathbf{m}(p, \nu) = \bar{n} \times \left( \int_{-\epsilon}^{+\epsilon} g \sigma^{\alpha \gamma} \xi d\xi + \int_{-\epsilon}^{+\epsilon} g \sigma^{\alpha \gamma} \xi^2 d\xi L_\beta^\gamma \right) \bar{\partial}_\gamma \nu_\alpha \quad (3.110)
\]

where we can finally define two tensors $N$ and $M$

\[
N = N^{\alpha \beta} (\bar{\partial}_\alpha \otimes \bar{\partial}_\beta) + N^{\alpha \xi} (\bar{\partial}_\alpha \otimes \bar{n}) \quad (3.111)
\]
\[
M = M^{\alpha \beta} (\bar{\partial}_\alpha \otimes \bar{\partial}_\beta) \quad (3.112)
\]

respectively as

\[
N^{\alpha \beta} = \int_{-\epsilon}^{+\epsilon} g \sigma^{\alpha \beta} d\xi + \int_{-\epsilon}^{+\epsilon} g \sigma^{\alpha \gamma} \xi d\xi L_\beta^\gamma \quad (3.113)
\]
\[
N^{\alpha \xi} = \int_{-\epsilon}^{+\epsilon} g \sigma^{\alpha \xi} d\xi \quad (3.114)
\]

and

\[
M^{\alpha \beta} = \int_{-\epsilon}^{+\epsilon} g \sigma^{\alpha \beta} \xi d\xi + \int_{-\epsilon}^{+\epsilon} g \sigma^{\alpha \gamma} \xi^2 d\xi L_\beta^\gamma \quad (3.115)
\]

such as

\[
\mathbf{n}(p, \nu) = N_{\nu} = N^{\alpha \beta} \nu_\alpha \bar{\partial}_\beta + N^{\alpha \xi} \nu_\alpha \bar{n} \quad (3.116)
\]
\[
\mathbf{m}(p, \nu) = \bar{n} \times M_{\nu} = \bar{n} \times M^{\alpha \beta} \nu_\alpha \bar{\partial}_\beta \quad (3.117)
\]
Two fields \( n \) and \( m \) are called *surface stress vector* and *surface couple vector* respectively, while the fields \( N \) and \( M \) are termed *surface stress tensor* and *surface couple tensor*.

From the above results it is immediate to notice that the surface stress vector \( n \) belongs to \( T_{Q} E \), consequently it can be split into a superficial part and an orthogonal part as follows

\[
    n = n^\parallel + n^\perp \tag{3.118}
\]

where

\[
    n^\parallel = N^{\alpha\beta} \nu_\alpha \tilde{\partial}_\beta \tag{3.119}
\]

\[
    n^\perp = N^{\alpha\xi} \nu_\alpha \tilde{n} \tag{3.120}
\]

while the surface couple vector \( m \) belongs to \( T_{Q} \) so that

\[
    m = m^\parallel \tag{3.121}
\]

As the last remark we point out that the coefficient \( g \) involved in the integration of Cauchy stress tensor along the thickness depends only on the geometrical features of the mid-surface \( Q \), in fact it is easy to prove the following expression

\[
    g = \det \left( S^h_i \right) = 1 + \xi H + \xi^2 K \tag{3.122}
\]

where \( H \) and \( K \) are the *mean curvature* and the *total curvature* of the surface \( Q \) defined in equations (1.160) and (1.159).

### 3.7.3 Body forces and load density

Suppose the the curve \( c : I \rightarrow R \rightarrow Q \) is closed in such a way as to capture a surface portion \( Q' \subset Q \) bounded by \( \partial Q \equiv c \). Assuming \( c \) to be a directrix, that is a curve through which a line generating a given ruled surface always passes, the generatrices directed along \( \tilde{n} \) define a cylinder \( G_c(\epsilon) \subset G(\epsilon) \) with thickness \( 2\epsilon \) and also bounded by the surface \( Q_c \cup Q^\epsilon \cup Q_{-\epsilon} \).

We assume that the volume forces acting at every point belonging to the cylinder \( G_c(\epsilon) \) and the load density acting at every point on the upper and lower surfaces \( Q^\epsilon \) and \( Q_{-\epsilon} \) can be integrated along
the thickness to yield a new force system defined on the mid–surface \( Q' \) as follows

\[
\bar{q} : Q' \rightarrow T_{Q'} E
\]

\[
\bar{s} : Q' \rightarrow TQ'
\]

(3.123) (3.124)

where \( \bar{q} = q^\beta \bar{\partial}_\beta + q^\xi \bar{n} \) represents the load vector and \( \bar{s} = \bar{n} \times s^\beta \bar{\partial}_\beta \) represents the load–moment vector.


### 3.7.4 Eulero’s equations

The equilibrium equations for the mid surface portion \( Q' \) can be written as follows

\[
\int_{\partial Q'} n(p, \nu) dl + \int_{Q'} \bar{q} dQ' = 0
\]

(3.125)

\[
\int_{\partial Q'} (m(p, \nu) + \bar{r} \times n(p, \nu)) dl + \int_{Q'} (\bar{r} \times \bar{q} + \bar{s}) dQ' = 0
\]

(3.126)

which yield

\[
\int_{\partial Q'} N \nu dl + \int_{Q'} \bar{q} dQ' = 0
\]

(3.127)

\[
\int_{\partial Q'} (\bar{n} \times M \nu + \bar{r} \times N \nu) dl + \int_{Q'} (\bar{r} \times \bar{q} + \bar{s}) dQ' = 0
\]

(3.128)

Making use of the divergence theorem enounced in equation (1.145) on page 29, and due to the arbitrariness of \( Q' \), the above equations become

\[
\text{div} N + \bar{q} = 0
\]

(3.129)

\[
\text{div}(\bar{n} \times M^{\alpha h} \bar{\partial}_h + \bar{r} \times N^{\alpha h} \bar{\partial}_h) + \bar{r} \times \bar{q} + \bar{s} = 0
\]

(3.130)

Equations (3.129) and (3.130) can be written in components as follows

\[
\nabla^\dagger_{\alpha} N^{\alpha\beta} + L^\beta_{\alpha} N^{\alpha\xi} + q^{\beta} = 0
\]

(3.131)

\[
\nabla_{\alpha} N^{\alpha\xi} + L_{\alpha\gamma} N^{\alpha\gamma} + q^{\xi} = 0
\]

(3.132)

\[
\nabla^\dagger_{\alpha} M^{\beta\alpha} - N^{\xi\beta} + s^{\beta} = 0
\]

(3.133)

\[
\eta_{\alpha\beta} \left( L^\alpha_{\gamma} M^{\beta\gamma} - N^{\alpha\beta} \right) = 0
\]

(3.134)
where equations (3.131) assure the translational equilibrium in the
tangent plane, while (3.132) represents the translational equilibrium
along the normal direction. Next, two equations in (3.133) impose
the rotational equilibrium about the surface axes, respectively. Fi-
ally, the last equilibrium condition (3.134) gives the symmetry to
the tensor $L^\alpha_\gamma M^{\beta\gamma} - N^{\alpha\beta}$.

**Proof**

Here we want to show all steps we made to pass from the equilib-
rium equations (3.129) and (3.130) to the corresponding expressions in
components (3.131) to (3.134).

Let us start from equation (3.129). We invoke the definition of di-
vergence for second order contravariant tensors already used in equation
(1.147), so we have

$$(\text{div} N)^h = N^\alpha_{\alpha} + \Gamma^\alpha_{\alpha\gamma} N^{\gamma h} + \Gamma^h_{\alpha} N^{\alpha h} =$$

$$= N^\alpha_{\alpha} + N^{\alpha\xi} + \Gamma^\alpha_{\alpha\gamma} N^{\gamma\beta} + \Gamma^\beta_{\alpha} N^{\alpha\xi} + \Gamma^\xi_{\alpha} N^{\alpha\gamma} + \Gamma^\gamma_{\alpha} N^{\alpha\xi} +$$

$$+ \Gamma^\beta_{\alpha\gamma} N^{\alpha\xi} + \Gamma^\beta_{\alpha\xi} N^{\alpha\gamma} + \Gamma^\xi_{\alpha\gamma} N^{\alpha\xi}$$

Now we just need to separate the tangential and normal components
as follows

$$(\text{div} N)^\beta = N^\alpha_{\alpha} + \Gamma^\alpha_{\alpha\gamma} N^{\gamma\beta} + \Gamma^\beta_{\alpha} N^{\alpha\xi} + \Gamma^\xi_{\alpha} N^{\alpha\gamma}$$ \hspace{1cm} (3.135)

$$(\text{div} N)^\xi = N^\alpha_{\alpha} + \Gamma^\alpha_{\alpha\gamma} N^{\gamma\xi} + \Gamma^\xi_{\alpha} N^{\alpha\gamma} + \Gamma^\gamma_{\alpha} N^{\alpha\xi}$$ \hspace{1cm} (3.136)

By virtue of the the identity $(\nabla_{\alpha} \mathbf{n})^\beta = L^\beta_{\alpha} = \Gamma^\beta_{\alpha}$ equation (3.135)
becomes

$$(\text{div} N)^\beta = \nabla^\dagger_{\alpha} N^\alpha_{\beta} + L^\beta_{\alpha} N^{\alpha\xi}$$ \hspace{1cm} (3.137)

where we have just collected the surface divergence terms into

$\nabla^\dagger_{\alpha} N^\alpha_{\beta} = N^\alpha_{\alpha\beta} + \Gamma^\alpha_{\alpha\gamma} N^{\gamma\beta} + \Gamma^\beta_{\alpha} N^{\alpha\gamma}$$ \hspace{1cm} (3.138)

Equation (3.137) proves the in–plane translational equilibrium ex-
pressed in (3.131).

Concerning equation (3.136), the translational equilibrium along
the normal direction is readily proved remembering both $\Gamma^\xi_{\alpha\gamma} = L_{\alpha\gamma}$ and

$\nabla_{\alpha} N^{\alpha\xi} = N^{\alpha\xi} + \Gamma^{\alpha}_{\alpha\gamma} N^{\gamma\xi} + \Gamma^{\xi}_{\alpha\gamma} N^{\alpha\xi}$$ \hspace{1cm} (3.139)$

---

8In literature the divergence of the surface tensor $N^\alpha_{\beta}$ is often denoted by $N^\alpha_{\beta}$.

9In literature the divergence $\nabla^\dagger_{\alpha} N^{\alpha\xi}$ is often denoted by $N^{\alpha\xi}$. 
Hence we obtain

\[(\text{div} N)^\xi = \nabla_\alpha N^{\alpha \xi} + L_{\alpha \gamma} N^{\alpha \gamma}\] (3.140)

which finally proves equation (3.132).

In order to prove equations (3.133) and (3.134), first we simplify equation (3.130) by taking into account equation (3.129). So it becomes

\[
\text{div} \tilde{n} \times M^{\alpha h} \bar{\partial}_h + \tilde{n} \times \text{div} \left( M^{\alpha h} \bar{\partial}_h \right) + \text{div} \bar{r} \times N^{\alpha h} \bar{\partial}_h + \bar{s} = 0 \] (3.141)

We can split the divergence of the tensor \(M^{\alpha h}\) in accordance with the results in (3.137) and (3.140), thus we have

\[
\begin{align*}
\nabla_\alpha \tilde{n} \times M^{\alpha h} \bar{\partial}_h + \tilde{n} \times \left( \nabla^\dagger_\alpha M^{\alpha \beta} + L^\beta_\gamma M^{\gamma \xi} \right) \bar{\partial}_\beta + \\
+ \tilde{n} \times \left( \nabla^\dagger_\alpha M^{\alpha \xi} + L_{\alpha \gamma} M^{\alpha \gamma} \right) \tilde{n} + \bar{r}_\alpha \times N^{\alpha h} \bar{\partial}_h + \bar{s} = 0
\end{align*}
\] (3.142)

which after further algebra becomes

\[
\begin{align*}
L^\gamma_\alpha \bar{\partial}_\gamma \times M^{\alpha \omega} \bar{\partial}_\omega + L^\gamma_\alpha \bar{\partial}_\gamma \times M^{\alpha \xi} \tilde{n} + \tilde{n} \times \left( \nabla^\dagger_\alpha M^{\alpha \beta} + L^\beta_\gamma M^{\gamma \xi} \right) \bar{\partial}_\beta + \\
+ \bar{\partial}_\alpha \times N^{\alpha \omega} \bar{\partial}_\omega + \bar{\partial}_\alpha \times N^{\alpha \xi} \tilde{n}_\omega + \tilde{n} \times s^\beta \bar{\partial}_\beta = 0
\end{align*}
\] (3.143)

Collecting the normal and tangential terms we obtain the following three scalar equations

\[
\eta_{\gamma \omega} \left( L^\gamma_\alpha M^{\alpha \omega} + N^{\gamma \omega} \right) = 0 \] (3.144)

and

\[
\tilde{n} \times \left( \nabla^\dagger_\alpha M^{\alpha \beta} - N^{\beta \xi} + s^\beta \right) \bar{\partial}_\beta = 0 \] (3.145)

which finally proves the rotational equilibrium (3.133) about the surface axes.

Usually a new variable is introduced to make easier possible further calculations; in fact we define the \textbf{pseudo-stress tensor} the symmetric tensor

\[
\tilde{N}^{\alpha \beta} = N^{\alpha \beta} - L^\alpha_\gamma M^{\beta \gamma} \] (3.146)

It is straightforward to notice that \(\tilde{N} \equiv N\) only when either a membrane stress state holds or for flat shells, namely when Wein-garten’s tensor is identically zero.
3.7.5 Membrane state of stress

In this last section we introduce an hypothesis on the state of the stress that enables us to derive a closed form solution for several shell geometries without invoking the constitutive law. Examples of these closed form solutions will be provided in appendix A.

A shell continuum is subjected to a membrane stress state when both the following condition hold

\[ N^\alpha\xi = 0 \]  \hspace{1cm} (3.147)
\[ M^{\alpha\beta} = 0 \]  \hspace{1cm} (3.148)

Hence, the equilibrium equations become

\[ \nabla_\alpha N^{\alpha\beta} + q^\beta = 0 \]  \hspace{1cm} (3.149)
\[ L_{\alpha\gamma} N^{\alpha\gamma} + q^\xi = 0 \]  \hspace{1cm} (3.150)
\[ \eta_{\alpha\beta} N^{\alpha\beta} = 0 \]  \hspace{1cm} (3.151)

where equation (3.149) represents the translational equilibrium along the tangent plane; equation (3.150) represents the equilibrium along \( \bar{n} \) and finally equation (3.151) states the rotational equilibrium about \( \bar{n} \) and establishes the symmetry of \( N \).
Chapter 4
Equations of elasticity

Chapters 2 and 3 of these notes do not specifically concern with the elastic media, in fact they can be understood for a generic continuum and studied independently. In this section we shall combine the previous results in order to investigate the response of elastic bodies under the action of forces.

A body is called elastic if it has the property of recovering its original shape when the forces which produce the deformations are removed. This property can be characterized mathematically by certain relationships connecting force and displacement, that are also called constitutive laws. In particular we will analyze the linear constitutive law as a generalization of the Hooke’s law.

4.1 The material law

It was Robert Hooke\textsuperscript{1} who in 1676 gave the first rough law of proportionality between forces and displacements for an elastic body. In order to understand the key features of elasticity, let us consider a thin rod with an initial cross section $A_0$, which is subjected to a variable tensile force $F$. We suppose that the stress is distributed uniformly over the area $A_0$ and the initial cross-sectional area stays constant. The stress is obtained by dividing the force at any stage by the area $A_0$. So, $\sigma = F/A_0$. The relationship between $F$ and the axial strain $\varepsilon$ is plotted in figure 4.1 on the next page.

Figure 4.1 shows that until the point $P$ the relationship $\sigma - \varepsilon$

\textsuperscript{1}Robert Hooke (July 18, 1635 Freshwater (Isle of Wight) - March 3, 1703 London) was an English scientist.

Source: http://turnbull.mcs.st-and.ac.uk/history/Biographies/Hooke.html.
is nearly a straight line with the following equation
\[ \sigma = E \varepsilon \quad (4.1) \]
where the constant of proportionality \( E \) is known as *modulus of elasticity* or *Young’s modulus*.

The greatest stress that can be applied to the rod without producing a permanent deformation is called *elastic limit* of the material. When the force \( F \) is increased beyond this limit the material goes in the elastic-plastic field. Namely, firstly the material reaches the *yield-point* \( Y \) at which the rod suddenly stretches, then the material reaches the *ultimate stress* at \( U \) where it offers the maximum stress. If the elongation increases again both the cross sectional area \( A_0 \) and the stress decrease until the rod breaks at \( B \).

From now on we shall study only the *elastic range*.

### 4.1.1 Generalized Hooke’s law

Here we want to extend the results of Hooke’s law to a multidimensional state of stress and strain. So, in accordance with equation (4.1), let us write a linear relation
\[ \sigma_{ij} = C_{ijhk} \varepsilon_{hk} \quad i, j, h, k = 1, 2, 3 \quad (4.2) \]

The coefficients \( C_{ijhk} \) are independent from the position of the reference point in the continuous medium, in other words we require the homogeneity of the body, that means uniformity in structure and composition. It can also be shown that the elastic constants
$C_{ijhk}$ are 81 components of a fourth order tensor which is termed \textit{elasticity tensor}.

Since the stress tensor $\sigma_{ij}$ is symmetric, an interchange of the first two indices in (4.2) does not alter its meaning. In addition to that, the symmetry of the strain tensor ensures also the symmetry of the last two indices, so that

$$C_{ijhk} = C_{jihk} \quad (4.3)$$

$$C_{ijhk} = C_{ijkh} \quad (4.4)$$

That means that the $3^4$ components of $C$ reduce to 36 independent constants. Let us show the expansion of a generic component of the stress tensor, that is

$$\sigma_{11} = C_{1111} \varepsilon_{11} + C_{1112} \varepsilon_{12} + C_{1113} \varepsilon_{13} + C_{1121} \varepsilon_{21} + C_{1122} \varepsilon_{22} + C_{1123} \varepsilon_{23} + 2C_{1112} \varepsilon_{12} + 2C_{1113} \varepsilon_{13} + 2C_{1123} \varepsilon_{23}$$

Equations (4.3) and (4.4) allow (4.5) to be rewritten as follows

$$\sigma_{11} = C_{1111} \varepsilon_{11} + C_{1122} \varepsilon_{22} + C_{1133} \varepsilon_{33} + 2C_{1112} \varepsilon_{12} + 2C_{1113} \varepsilon_{13} + 2C_{1123} \varepsilon_{23}$$

Thus, the whole elastic matrix can be written as

$$\begin{pmatrix} 
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{31}
\end{pmatrix} = \begin{pmatrix}
C_{1111} & C_{1112} & C_{1113} & 2C_{1112} & 2C_{1113} & 2C_{1123} & 2C_{1131} \\
C_{2222} & C_{2233} & 2C_{2212} & 2C_{2223} & 2C_{2231} & 2C_{2323} & 2C_{2331} \\
C_{3333} & C_{3312} & 2C_{3323} & 2C_{3331} & 2C_{3123} & 2C_{3131} & 2C_{3231} \\
\text{sym.} & 2C_{1212} & 2C_{1223} & 2C_{1231} & \text{sym.} & 2C_{3231} & \text{sym.} \\
\text{sym.} & 2C_{2323} & 2C_{2331} & \text{sym.} & 2C_{3131} & \text{sym.} & \text{sym.}
\end{pmatrix} \begin{pmatrix} 
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{12} \\
\varepsilon_{23} \\
\varepsilon_{31}
\end{pmatrix}$$

which, making use of the symmetry relationships expressed in (4.3) and (4.4), simplifies as follows

$$\begin{pmatrix} 
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{31}
\end{pmatrix} = \begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & \text{sym.} \\
c_{33} & c_{34} & c_{35} & c_{36} & \text{sym.} & \text{sym.} \\
\text{sym.} & c_{44} & c_{45} & c_{46} & \text{sym.} & \text{sym.} \\
\text{sym.} & \text{sym.} & \text{sym.} & \text{sym.} & \text{sym.} & \text{sym.}
\end{pmatrix} \begin{pmatrix} 
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{12} \\
\varepsilon_{23} \\
\varepsilon_{31}
\end{pmatrix}$$
Later on, see equation (6.10), we will also introduce another symmetry condition that has been assumed in the above. Namely, the condition

$$C_{ijhk} = C_{hkij}$$  \hspace{1cm} (4.6)

that further reduces the independent elastic constant from 36 to 21. So, the latter material equation represents the constitutive law for an \textit{anisotropic} elastic material. However, most of the engineering materials have some symmetry properties which allow further reductions of the elastic constants.

The highest degree of symmetry leads to the so called \textit{isotropic} material. We define an isotropic material an elastic continuum which has the same response in any direction, so that the elastic tensor is not influenced by any rotation of the references axes.

Let the elastic tensor be represented by $C_{ijhk}$ with respect to the cartesian coordinate $\{x^i\}$ whose basis is $B = \{\vec{e}_i\}$. With respect to a rotated system $\{x'^i\}$ with basis $B' = \{\vec{e}'_i\}$ the elasticity tensor is $C'_{ijhk}$. By the definition of isotropic material, we expect that the elasticity tensor does not change. In order to show this, let us recall the transformation relations (1.36) on chapter 1. Here we are dealing with a Cartesian coordinate system, hence it does not matter if the indices are all subscripts. So, we have

$$C'_{ijhk} = a'_i a'_j a'_m a'_{hn} C_{lmno} a'_{oh} a'_{nk}$$

$$= a'_i a'_j a'_m a'_{hn} a'_{ko} a'_{kn} C_{lmno}$$ \hspace{1cm} (4.7)

but to ensure the immunity against the rotation of the reference system, we impose

$$C'_{ijhk} = C_{lmno}$$ \hspace{1cm} (4.8)

that is only satisfied if the elasticity tensor assumes the following form

$$C_{lmno} = \lambda \delta_{ln} \delta_{no} + \mu \delta_{ln} \delta_{mo} + \kappa \delta_{lo} \delta_{mn}$$ \hspace{1cm} (4.9)

where $\lambda, \mu, \kappa$ are elastic constants$^2$.

$^2$This can be proved by replacing equation (4.9) into (4.7), as follows

$$C'_{ijhk} = a'_i a'_j a'_m a_{'hn} a'_{'ko} (\lambda \delta_{lm} \delta_{no} + \mu \delta_{ln} \delta_{mo} + \kappa \delta_{lo} \delta_{mn}) =$$

$$= \lambda a'_i a'_j a_{'hn} a'_{'ko} + \mu a'_i a'_j a'_{'hn} a'_{'ko} + \kappa a'_i a'_j a_{'hn} a'_{'ko} =$$

$$\lambda \delta_{ih} \delta_{jk} + \mu \delta_{ij} \delta_{hk} + \kappa \delta_{ik} \delta_{jh}$$

that is exactly the expression (4.9). Note that we have used the identity $a'_{ps} a'_{qs} = \delta_{pq}$ provided by equations (1.21) and (1.24) on page 7.
In equations (4.3) and (4.4) we have already noticed the symmetry of \( C \) in relation to the two front and two back indices, let us show now that one more reduction is possible

\[
\begin{align*}
C_{ijhk} &= \lambda \delta_{ij}\delta_{hk} + \mu \delta_{ih}\delta_{jk} + \kappa \delta_{ik}\delta_{jh} \quad (4.10) \\
C_{ijkh} &= \lambda \delta_{ij}\delta_{kh} + \mu \delta_{ik}\delta_{jh} + \kappa \delta_{ih}\delta_{jk} \quad (4.11)
\end{align*}
\]

where, subtracting term by term and considering the symmetry of the unit tensor \( \delta_{ij} \), equations (4.10) and (4.11) lead to the only possible condition

\[
\begin{align*}
\mu (\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}) + \kappa (\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk}) &= 0 \Rightarrow \\
\mu (\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}) - \kappa (\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}) &= 0 \Rightarrow \\
(\mu - \kappa) (\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}) &= 0 \quad (4.12)
\end{align*}
\]

which is only true if \( (\mu - \kappa) = 0 \). So, the relationship between \( \kappa \) and \( \mu \) further reduces the number of elastic constants to 2. Namely, we have

\[
C_{ijhk} = \lambda \delta_{ij}\delta_{hk} + \mu (\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}) \quad (4.13)
\]

The Hooke’s law becomes

\[
\begin{align*}
\sigma_{ij} &= C_{ijhk}\varepsilon_{hk} = \lambda \delta_{ij}\delta_{hk}\varepsilon_{hk} + \mu (\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}) \varepsilon_{hk} = \\
&= \ldots \\
&= \lambda \delta_{ij}\varepsilon_{hh} + 2\mu \varepsilon_{ij} \quad (4.14)
\end{align*}
\]

where we have used \( \delta_{hk}\varepsilon_{hk} = \varepsilon_{hh} = \text{tr} \varepsilon_{hk} \).

Equation (4.14) is the generalized form of Hooke’s law, valid for homogeneous, isotropic, elastic bodies. \( \lambda \) and \( \mu \) are called Lamé constants\(^3\).

\(^3\)Gabriel Lamé (July 22, 1795 Tours - May 1, 1870 Paris) was a French mathematician and engineer.

Source: http://turnbull.mcs.st-and.ac.uk/history/Mathematicians/Lame.html.
The trace of the stress tensor is readily computed by contracting the indices, so that

\[ \sigma_{ii} = 3\lambda \varepsilon_{hh} + 2\mu \varepsilon_{ii} \Rightarrow \]  
\[ \sigma_{ii} = (2\mu + 3\lambda) \varepsilon_{hh} \Rightarrow \]  
\[ \varepsilon_{hh} = \frac{\sigma_{ii}}{(2\mu + 3\lambda)} \]  
(4.17)

where we can put \( \text{tr} \sigma_{ij} = \sigma_{ii} = \Sigma \) and \( \text{tr} \varepsilon_{ij} = \varepsilon_{ii} = \Theta \).

The above expression (4.17) is useful if we solve (4.14) for \( \varepsilon_{ij} \). In fact, we have

\[ \varepsilon_{ij} = \frac{1}{2\mu} \sigma_{ij} - \frac{\lambda}{2\mu} \delta_{ij} \Theta \]  
(4.18)

and in observance of (4.17) we obtain

\[ \varepsilon_{ij} = \frac{1}{2\mu} \sigma_{ij} - \frac{\lambda}{2\mu (3\lambda + 2\mu)} \delta_{ij} \Sigma \]  
(4.19)

Now, let us consider an axial state of stress. The stress tensor is

\[ \sigma_{ij} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]  
form (4.19) we have

\[ \varepsilon_{11} = \frac{1}{2\mu} \left( 1 - \frac{\lambda}{(3\lambda + 2\mu)} \right) \sigma_{11} = \]  
\[ = \ldots \]  
\[ = \frac{\lambda - \mu}{\mu (3\lambda + 2\mu)} \sigma_{11} \]  
(4.20)

\[ \varepsilon_{22} = \varepsilon_{33} = -\frac{\lambda}{2\mu (3\lambda + 2\mu)} \sigma_{11} \]  
(4.21)

Let us define \textit{Poisson's ratio} \( \nu \) as follows

\[ \nu = -\frac{\varepsilon_{11}}{\varepsilon_{22}} = -\frac{\varepsilon_{11}}{\varepsilon_{33}} = \frac{\lambda}{2(\mu + \lambda)} \]  
(4.23)
According to Hooke’s law in the original form, see equation (4.1), we can see that
\[
\frac{1}{E} = \frac{\lambda - \mu}{\mu (3\lambda + 2\mu)} \Rightarrow E = \frac{\mu (3\lambda + 2\mu)}{\lambda - \mu} \quad (4.24)
\]

So, we have proved that Lamé constants can be replaced by \( E \) and \( \nu \) which lead to writing the alternative expressions of the constitutive law

\[
\varepsilon_{ij} = \frac{1}{E} \left( (1 + \nu) \sigma_{ij} - \nu \delta_{ij} \Sigma \right) \quad (4.25)
\]

\[
\sigma_{ij} = \frac{E}{1 + \nu} \left( \varepsilon_{ij} + \frac{\nu}{1 - 2\nu} \delta_{ij} \Theta \right) \quad (4.26)
\]

Table 4.1 shows the relationships between elastic constants.

### 4.2 The linear elastic problem

In this section we are going to sum up equations and unknown quantities which define the classical linear elastic problem. Then we will estimate the distribution of stresses and strain as well as displacements at all points of the body when certain boundary conditions are given. Let us balance the unknowns and the equations, we have fifteen unknowns (6 stress components + 6 strain components + 3 displacement components) for all points in the continuous and just fifteen equations (6 equilibrium + 6 compatibility + 3

<table>
<thead>
<tr>
<th>( \lambda, \mu )</th>
<th>( \lambda )</th>
<th>( \mu )</th>
<th>( E )</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>-</td>
<td>-</td>
<td>( \frac{\mu (3\lambda + 2\mu)}{\lambda + \mu} )</td>
<td>( \frac{\lambda}{2(\lambda + \mu)} )</td>
</tr>
<tr>
<td>( \lambda, \nu )</td>
<td>-</td>
<td>( \frac{\lambda (1 - 2\nu)}{2\nu} )</td>
<td>( \frac{\lambda (1 + \nu) (1 - 2\nu)}{\nu} )</td>
<td>-</td>
</tr>
<tr>
<td>( \mu, E )</td>
<td>( \frac{\mu (E - 2\mu)}{3\mu - E} )</td>
<td>-</td>
<td>-</td>
<td>( \frac{E - 2\mu}{2\mu} )</td>
</tr>
<tr>
<td>( \mu, \nu )</td>
<td>( \frac{2\mu \nu}{1 - 2\nu} )</td>
<td>-</td>
<td>( 2\mu (1 + \nu) )</td>
<td>-</td>
</tr>
<tr>
<td>( E, \nu )</td>
<td>( \frac{E \nu}{(1 + \nu)(1 - 2\nu)} )</td>
<td>( \frac{E}{2(1 + \nu)} )</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4.1: Relationships between the main elastic constants.
boundary conditions). So, for a given linear elastic body $V$ we have

\begin{align*}
C &= \text{const.} \quad (4.27) \\
\bar{b} &= \bar{b}(p) \quad \forall p \in V \quad (4.28) \\
\bar{f} &= \hat{f}(p) \quad \forall p \in S_\sigma \quad (4.29) \\
\bar{u} &= \hat{u}(p) \quad \forall p \in S_u \quad (4.30)
\end{align*}

In order to solve the linear elastic problem we start from the known quantities (4.27) to (4.30), and through the following available equations

- compatibility equations
  \[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad \text{on } V \quad (4.31) \]

- equilibrium equations
  \[ \sigma_{ij,j} + b_i = 0 \quad \text{on } V \quad (4.32) \]

- constitutive laws
  \[ \sigma_{ij} = \frac{E}{1 + \nu} \left( \varepsilon_{ij} + \frac{\nu}{1 - 2\nu} \delta_{ij} \varepsilon_{ij} \right) \quad \text{on } V \quad (4.33) \]

- boundary conditions
  \[ \sigma_{ij} n_j = \hat{f}_i \quad \text{on } S_\sigma \quad (4.34) \]
  \[ u_i = \hat{u}_i \quad \text{on } S_u \quad (4.35) \]

we will formulate two boundary-value problems.

### 4.2.1 Boundary value problem in terms of stresses

This first boundary value problem can be stated as follows:

*Determine the distribution of stresses and displacements in the interior of an elastic body in equilibrium when the body forces are prescribed and the distribution of the forces acting on the surface of the body is known*

\footnote{Sokolnikoff [1].}
Following the above formulation, the procedure for solving the problem would suggest writing the available equations entirely in terms of stress. To this aim let us start from equation (2.75)

\[
\varepsilon_{ij,hk} + \varepsilon_{hk,ij} - \varepsilon_{ih,jk} - \varepsilon_{jk,ih} = 0
\]

and consider the constitutive law (4.25), so that

\[
\frac{1 + \nu}{E} (\sigma_{ij,hk} + \sigma_{hk,ij} - \sigma_{ih,jk} - \sigma_{jk,ih}) = \frac{\nu}{E} (\delta_{ij}\sigma_{nn,hk} + \delta_{hk}\sigma_{nn,ij} - \delta_{ih}\sigma_{nn,jk} - \delta_{jk}\sigma_{nn,ih})
\]

Equation (4.37) represents a set \(3^4 = 81\) equations since all the four indices \(i,j,h,k\) run from 1 to 3. Not all of these equations are independent, indeed the system (4.37) contains only 6 independent equations. A first reduction of equations is due to the contraction \(h = k\) that yields

\[
\sigma_{ij, kk} + \sigma_{kk, ij} - \sigma_{ik,jk} - \sigma_{jk,ik} = \frac{\nu}{1 + \nu} (\delta_{ij}\sigma_{nn, kk} + \delta_{kk}\sigma_{nn, ij} - \delta_{ih}\sigma_{nn, jk} - \delta_{jk}\sigma_{nn, ik})
\]

that, by denoting \(\Sigma = \text{tr}\sigma_{ij} = \sigma_{ii}\) and \(\sigma_{ij, kk} = \nabla^2\sigma_{ij}\), becomes

\[
\nabla^2\sigma_{ij} + \Sigma,ij - \sigma_{ik,jk} - \sigma_{jk,ik} = \frac{\nu}{1 + \nu} (\delta_{ij}\nabla^2\Sigma + \nabla^2\Sigma_{ij})
\]

By virtue of the equilibrium equations (4.32), the above expression can be rewritten as follows

\[
\nabla^2\sigma_{ij} + \frac{1}{1 + \nu} \Sigma,ij = - \left( b_{i,j} + b_{j,i} - \frac{\nu}{1 + \nu} \delta_{ij}\nabla^2\Sigma \right)
\]

which is a set of 6 independent equations.

Next, in order to express \(\nabla^2\Sigma\) as a function of the body force \(\bar{b}\), we put \(h = i\) and \(k = j\) in equation (4.37), so that, after a bit of algebra, we have

\[
\sigma_{ij,ij} = \nabla^2\Sigma - 2\frac{\nu}{1 + \nu} \nabla^2\Sigma
\]

\[
= \ldots
\]

\[
= \frac{1 - \nu}{1 + \nu} \nabla^2\Sigma
\]

(4.41)
and finally, by invoking the derivative of the equilibrium equation that provides the relationships $b_{i,i} = \sigma_{ij,ij}$, we get

$$\nabla^2 \Sigma = -\frac{1 + \nu}{1 - \nu} b_{i,i} \tag{4.42}$$

Now, going back to equation (4.40) and making use of the latter result, it is not a difficult task to obtain the following expression

$$\nabla^2 \sigma_{ij} + \frac{1}{1 + \nu} \Sigma_{ij} = - \left( b_{i,j} + b_{j,i} + \frac{\nu}{1 - \nu} \delta_{ij} \text{div} \bar{b} \right) \tag{4.43}$$

Equations (4.43) were derived by Michell\textsuperscript{5} in 1900 and by Beltrami\textsuperscript{6} in the 1892 for the special case when the body forces are absent. Nevertheless, it is common to refer to equation (4.43) as Beltrami-Michell equations.

In case of missing or constant volume forces equation (4.43) assumes the straightforward form

$$\nabla^2 \sigma_{ij} + \frac{1}{1 + \nu} \Sigma_{ij} = 0 \tag{4.44}$$

### 4.2.2 Boundary value problem in terms of displacements

The second boundary value problem can be stated as follows:

*Determine the distribution of stresses and displacements in the interior of an elastic body in equilibrium*

\textsuperscript{5}John Henry Michell (October 26, 1863 - February 3, 1940) was an Australian mathematician.

\textsuperscript{6}Eugenio Beltrami (November 16, 1835 Cremona - February 18, 1900 Rome) was an Italian mathematician.
when the body forces are prescribed and the displacements of the points on the surface are prescribed functions\(^7\).

By replacing the constitutive law in the form of (4.14) into equilibrium equation, we obtain

\[(\lambda \delta_{ij} \varepsilon_{kk})_{,j} + 2\mu \varepsilon_{ij,j} + b_i = 0\] (4.45)

that is

\[\lambda \varepsilon_{kk,i} + 2\mu \varepsilon_{ij,j} + b_i = 0\] (4.46)

and in accordance with the compatibility equations we have

\[\lambda u_{k,ki} + \mu (u_{i,jj} + u_{j,ij}) + b_i = 0\] (4.47)

\[\lambda u_{k,ki} + \mu \nabla^2 u_i + \mu u_{k,ik} + b_i = 0\] (4.48)

\[(\lambda + \mu) u_{k,ki} + \mu \nabla^2 u_i + b_i = 0\] (4.49)

that in the vectorial form reads

\[(\lambda + \mu) \text{grad} \text{ div} \bar{u} + \mu \nabla^2 \bar{u} + \bar{b} = 0\] (4.50)

Equation (4.49) (or equivalently equation (4.50)) is called Lamé-Navier equation and together with the boundary conditions expressed by equation (4.35) define the boundary problem inn terms of displacements.

Once the first boundary value problem has been solved, i.e. when the displacements are known, the state of strain and hence the state of stress can be found though equations (4.31) and (4.33), respectively.

Further attention should be focused on the case when body forces do not occur or they are constant. First, consider the divergence of equation (4.49)

\[(\lambda + \mu) u_{k,ikki} + \mu \nabla^2 u_{i,i} + b_{i,i} = 0\] (4.51)

that yields

\[\lambda \nabla^2 u_{k,k} + 2\mu \nabla^2 u_{k,k} + b_{i,i} = (\lambda + 2\mu) \nabla^2 u_{k,k} + b_{i,i} = 0\] (4.52)

\(^7\)Sokolnikoff [1].
which, under the hypothesis of \( b_i = \text{const.} \), so that \( b_{i,i} = 0 \), gives

\[
\nabla^2 u_{k,k} = \nabla^2 \Theta = 0 \tag{4.53}
\]

where we have set \( \Theta = \text{tr} \epsilon_{ij} = \epsilon_{ii} \).

Moreover, recalling (4.17) it is also proved that

\[
\nabla^2 \sigma_{kk} = 0 \tag{4.54}
\]

We can finally say that if the volume forces are constant, the boundary linear elastic problem in terms of displacements turns into a general boundary values problem of a biharmonic differential equation.

### 4.3 Constitutive equation for shell continuums

The Kirchhoff–Love hypothesis and the inextensibility of material fibers along \( \bar{n} \) allows one to consider the shear stress components \( N^\xi \) unrelated to strains, so that the constitutive problem can be solved through the plane stress model. Thus, components \( N^\xi \) are found only by means of the equilibrium equations. The analytical derivation of the constitutive equations is beyond the scope of this book, so we will just present the final equations that will be used in the appendix A in order to solve some case studies. However, readers can find thorough discussions in [3] and [16].

Suppose a membrane state of stress, the constitutive equations are the following

\[
\tilde{N}^{\alpha\beta} = DH^{\alpha\beta\lambda\mu} \alpha_{\lambda\mu} \tag{4.55}
\]

\[
M^{\alpha\beta} = BH^{\alpha\beta\lambda\mu} \omega_{\lambda\mu} \tag{4.56}
\]

where

\[
H^{\alpha\beta\lambda\mu} = \frac{1}{2} \left( g^{\alpha\lambda} g^{\beta\mu} + g^{\alpha\mu} g^{\beta\lambda} + \frac{2\nu}{1-\nu} g^{\alpha\beta} g^{\lambda\mu} \right) \tag{4.57}
\]

The fourth–order tensor \( H^{\alpha\beta\lambda\mu} \) has the following symmetries

\[
H^{\alpha\beta\lambda\mu} = H^{\beta\alpha\lambda\mu} = H^{\alpha\beta\mu\lambda} = H^{\lambda\mu\alpha\beta} \tag{4.58}
\]
Finally, coefficients $D$ and $B$ are the in-plane and the bending stiffness, respectively, defined as

\[
D = \frac{E(2\varepsilon)}{1 - \nu^2} \tag{4.59}
\]

\[
B = \frac{E(2\varepsilon)^3}{12(1 - \nu^2)} \tag{4.60}
\]